

Debre Birhan University

College of Natural and Computational Science

Department of Physics

Reference material for Remedial program

Short reference material

April 2023

Chapter 1

Vectors

1.1 Vectors

Learning competencies

- Demonstrate an understanding of the difference between scalars and vectors and give common examples.
- Explain what a position vector is.
- Use vector notation and arrow representation of a vector.
- Specify the unit vector in the direction of a given vector.
- Determine the magnitude and direction of the resolution of two or more vectors using Pythagoras's theorem and trigonometry.
- Add vectors by graphical representation to determine a resultant.
- Add/subtract two or more vectors by the vector addition rule.
- Use the geometric definition of the scalar product to calculate the scalar product of two given vectors.
- Use the scalar product to determine projection of a vector onto another vector.
- Use the vector product to test for collinear and orthogonality vectors.
- Explain the use of knowledge of vectors in understanding natural phenomena.

1.1.1 Vector and Scalar quantities

Scalars are the physical quantities that have the only magnitude. **Examples** of scalars are electric charge, density, mass etc. Vectors are physical Quantities that must be described by both magnitude and direction.

Example: **Velocity, Force, Torque, Electric field** etc.

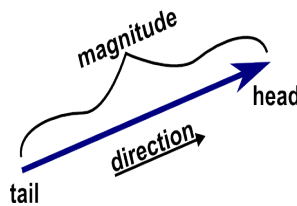
1.1.2 Vector Representation

Vectors are represented in two methods (**Analytical/Algebraic**) and **Graphical/Geometrical**)

1. **Analytical methods:** Vectors are represented analytically by a letter with an arrow over its head or with bold face letter.

Example: Force $\Rightarrow \vec{F}$ or **F**, Momentum $\Rightarrow \vec{P}$ or **P**, Vector A $\Rightarrow \vec{A}$ or **A**

2. **Graphical/Geometrical methods:** Graphically vectors are represented by a straight line and arrow drawn to the scale. The length of the line is the magnitude



of the vector and arrow tells us the direction.

1.1.3 Vector Addition and subsection

The sum of two or more vectors is called **resultant vector** ($\vec{R}$). Note that subtraction is addition of the negative ie

$$\vec{R} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

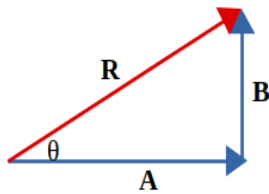
Vector addition is not simple algebraic addition of numbers that is handled with the normal rules of arithmetic. It Obeys the laws of vector addition as follows

- The resultant of two vectos having the same direction is algebraic sum of the two vectors with the same direction as both.

Example $\vec{A} = 8m$ East and $\vec{B} = 6m$ East then $\vec{R} = \vec{A} + \vec{B} = 14m$ East

- The resultant of two vectors having opposite direction has magnitude equal to the difference of magnitudes of the vectors and the resultant has the same direction as the larger vector. Example: $\vec{A} = 8m$ East and $\vec{B} = 6m$ West then $\vec{R} = \vec{A} + (-\vec{B}) = 2m$ East

- The resultant of two vectors acting at right angle with each other is obtained using Pythagorus theorem. Example: $\vec{A} = 8m$ East and $\vec{B} = 6m$ North then, And then the magnitude of $\vec{R}$ obtained using **Pythagorus theorem** as



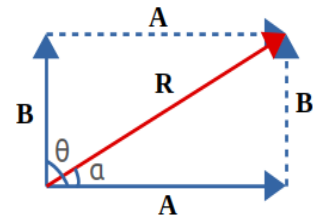
Triangular Methods:

Joining the vectors head to in a certain order, then the vector that is drawn from the tail of the first vector to the head of the last vector is resultant vector (R)

OR

Parallelogram methods

Dr-owing the vectors from the same origin and then complete the parallelogram. The resultant vector R is the diagonal of the parallelogram.



$$R^2 = A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 + 2AB\cos 90^\circ = A^2 + B^2$$

$$R = \sqrt{(8m)^2 + (6m)^2} = 10m$$

Direction of $\vec{R}$ obtained by **trigonometry**

$$\tan\theta = \frac{\text{Opp.}}{\text{adj.}} = \frac{|B|}{|A|}$$

$$\theta = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{6}{8}\right) = 36.87^\circ$$

- If the two vectors inclined at a certain anle θ to each other.

The magnitude of the vector $\vec{R} = \vec{A} + \vec{B}$ is given by

$$|\vec{R}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

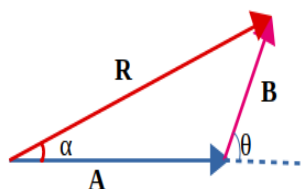
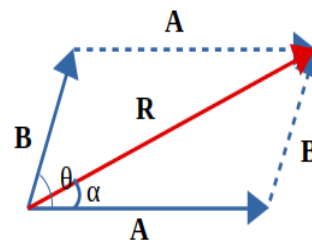
The magnitude of the vector $\vec{R} = \vec{A} - \vec{B}$ is given by

$$|\vec{R}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

And its direction is given by $\alpha = \tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)$

Parallelogram methods

Dr-owing the vectors from the same origin and then complete the parallelogram. The resultant vector R is the diagonal of the parallelogram.

Triangular Methods:

Joining the vectors head to in a certain order, then the vector that is drawn from the tail of the first vector to the head of the last vector is resultant vector (R)

Or using sin law $\frac{\sin(180^\circ - \theta)}{R} = \frac{\sin \alpha}{B}$ or $\sin \alpha = \frac{\sin(180^\circ - \theta)}{R} B$

$$\alpha = \sin^{-1}\left(B \frac{\sin(180^\circ - \theta)}{R}\right)$$

NB:

- If the vectors form a closed polygon when joined head to tail in a certain order, their resultant is **zero or null vector**
- Two or more vectors are equal if and only if they are
 - the same physical quantities
 - have the same magnitude and
 - have the direction

Class work

1. Given Vector $\vec{A}$ and $\vec{B}$. Find the resultant vector $\vec{R} = \vec{A} + \vec{B}$

- a) If the $\vec{A} = 4\text{unit East}$ and $\vec{B} = 3\text{unit East}$
- b) If the $\vec{A} = 4\text{unit East}$ and $\vec{B} = 3\text{unit West}$
- c) If the $\vec{A} = 4\text{unit East}$ and $\vec{B} = 3\text{unit north}$
- d) If the $\vec{A} = 4\text{unit East}$ and $\vec{B} = 3\text{unit at } 60^\circ \text{ north of east}$

2. A car travels 20.0km due north and then 35.0km in direction 60° west of north. Find the magnitude and direction of the car's resultant displacement. (**ans** $\vec{S} = 48.2\text{km}$ at 39.0° west of north).

1.1.4 Vector Components

Components of vectors are projection of vectors along coordinate axis (x, y, z-axis). This means splitting vector into its Components. Consider the following figures. From the figures we can see that A_x and A_y forms two sides of right angle triangle with hypotenuse of length A. Using simple trigonometry (definition of sin and cosin) we see that

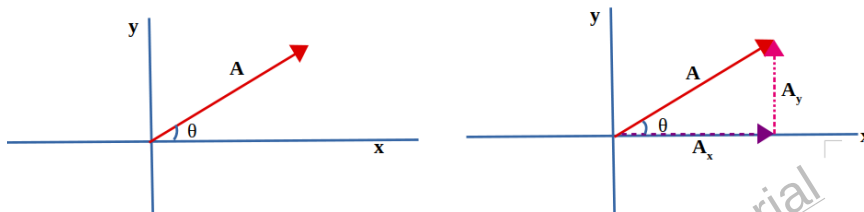


Figure 1.1: Vector Components

$$\cos\theta = \frac{A_x}{|\vec{A}|} \implies A_x = A\cos\theta$$

$$\sin\theta = \frac{A_y}{|\vec{A}|} \implies A_y = A\sin\theta$$

Thus $\vec{A} = \vec{A}_x + \vec{A}_y$ for three dimension $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$

1.1.5 Unit Vector

Unit vector is dimensionless vector with unit magnitude.

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

$\hat{A}$ read as A hat or caret is a unit vector that points in the direction of vector A. We shall use the symbols $\hat{i}$, $\hat{j}$ and $\hat{k}$ to represent a unit vector pointing in the positive x, y and z direction respectively as we can see from the figure above.

- The unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ in rectangular coordinate System. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are mutually perpendicular axes.

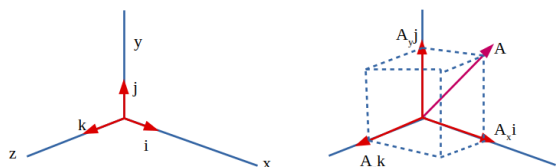


Figure 1.2: Unit Vector in Rectangular Coordinate Axis

- In general Vector A in rectangular coordinate system can be written as the sum of three vectors each which is parallel to a coordinate axes $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Addition and Subtraction of two vectors $\vec{A}$ and $\vec{B}$ can be written intermes of unit vector as

$$\begin{aligned}\vec{A} + \vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= (A_x\hat{i} + B_x\hat{i}) + (A_y\hat{j} + B_y\hat{j}) + (A_z\hat{k} + B_z\hat{k}) = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}\end{aligned}$$

Class work

- Given vectors $\vec{A} = 4m\hat{i} + 3m\hat{j}$, $\vec{B} = 2m\hat{i} - 3m\hat{j}$, $\vec{C} = 2m\hat{i} + 3m\hat{j} - 2m\hat{k}$ and $\vec{D} = 1m\hat{i} - 2m\hat{j} + 2m\hat{k}$. Find
 - $|\vec{A}|$
 - $2\vec{A} + \vec{B} - \vec{C}$
 - Unit vector in the direction of vector R such that $2\vec{C} + \vec{B} - \vec{R} = 0$
- A particle undergoes three consecutive displacements $\vec{d}_1 = (15\hat{i} + 30\hat{j} + 12\hat{k})cm$, $\vec{d}_2 = (23\hat{i} - 14\hat{j} - 5.0\hat{k})cm$, $\vec{d}_3 = (-13\hat{i} + 15\hat{j})cm$. Find
 - The components of the resultant displacement and its magnitude
 - Unit vector in the direction of resultant displacement

1.2 Multiplication of vectors

Vector multiplication refer to several operations between two (or more) vectors. It may concern any of the following articles:

- Scalar-vector multiplication
- Dot product
- Cross product

1.2.1 Scalar-vector multiplication

Multiplication of a vector by a scalar changes the magnitude of the vector, but leaves its direction unchanged. The scalar changes the size of the vector. The scalar "scales" the vector.

For example, If

$$\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Multiplied $\vec{A}$ by the scalar b is

$$b\vec{A} = b(a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) = ba_x \hat{i} + ba_y \hat{j} + ba_z \hat{k}$$

Scalar multiplication obeys the following rules:

- Additivity in the scalar: $(c + d)\vec{v} = c\vec{v} + d\vec{v}$;
- Additivity in the vector: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$;
- Compatibility of product of scalars with scalar multiplication: $(cd)\vec{v} = c(d\vec{v})$;
- Multiplying by 1 does not change a vector: $1\vec{v} = \vec{v}$;
- Multiplying by 0 gives the zero vector: $0\vec{v} = \vec{0}$;
- Multiplying by -1 gives the additive inverse: $(-1)\vec{v} = -\vec{v}$.

1.2.2 Dot product

The dot product of two vectors is the magnitude of one times the projection of the second onto the first. The symbol used to represent this operation is a small dot at middle height ($\cdot$), which is where the name "dot product" comes from. Since this product has magnitude only, it is also known as the scalar product. Mathematically defined as

$$\vec{A} \cdot \vec{B} = AB \cos(\theta)$$

where θ the angle between $\vec{A}$ and $\vec{B}$

Let

$$\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

and

$$\vec{B} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

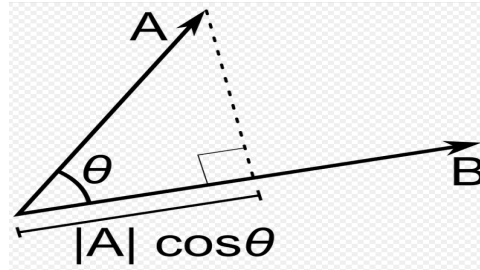


Figure 1.3: Dot Product

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \cdot (b_x \hat{i} + b_y \hat{j} + b_z \hat{k}) \\ &= a_x b_x \hat{i} \cdot \hat{i} + a_y b_y \hat{j} \cdot \hat{j} + a_z b_z \hat{k} \cdot \hat{k} \\ &= a_x b_x + a_y b_y + a_z b_z\end{aligned}$$

since $\hat{i} \cdot \hat{i} = (1)(1)\cos(0) = 1$, $\hat{j} \cdot \hat{j} = (1)(1)\cos(0) = 1$, $\hat{k} \cdot \hat{k} = (1)(1)\cos(0) = 1$ but $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = (1)(1)\cos(90^\circ) = 0$, $\hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = (1)(1)\cos(90^\circ) = 0$, $\hat{k} \cdot \hat{i} = \hat{i} \cdot \hat{k} = (1)(1)\cos(90^\circ) = 0$

Dot Product Properties of Vector:

- Dot product of two vectors is commutative i.e. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- If $\vec{A} \cdot \vec{B} = 0$ then it can be clearly seen that either $\vec{A}$ or $\vec{B}$ is zero or $\cos(\theta) = 0$.
- Also we know that using scalar product of vectors $(p\vec{A}) \cdot (q\vec{B}) = (p\vec{B}) \cdot (q\vec{A}) = pq(\vec{A} \cdot \vec{B})$
- The dot product of a vector to itself is the magnitude squared of the vector i.e. $\vec{A} \cdot \vec{A} = AA\cos(0) = A^2$
- Distributive Property: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- Non-Associative Property: $\vec{A} \cdot (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \cdot (\vec{A} \cdot \vec{C})$, because the dot product between a scalar and a vector is not allowed.

1.2.3 Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is vector $\vec{c}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and equal magnitude to the area of the parallelogram between $\vec{a}$ and $\vec{b}$. The symbol used to represent this operation is a large diagonal cross ($\times$), which is where the name "cross product" comes from. Since this product has magnitude and direction, it is also known as the **vector product**.

$$\vec{a} \times \vec{b} = ab \sin(\theta) \hat{n}$$

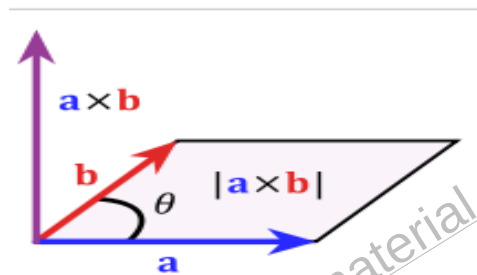


Figure 1.4: Cross Product

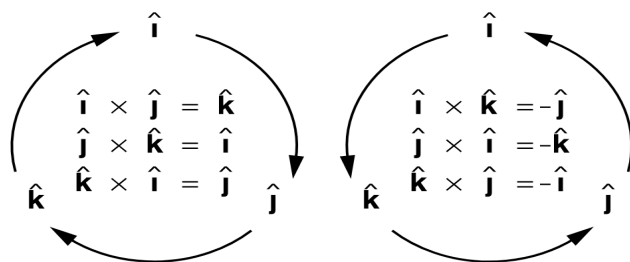
The vector $\hat{n}$ (n hat) is a unit vector perpendicular to the plane formed by the two vectors and θ is the angle between $\vec{a}$ and $\vec{b}$. The direction of $\hat{n}$ is determined by the right hand rule.

Cross Product Properties :

- the cross product is distributive: $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$
- the cross product is not commutative: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

$$\text{but } \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

- the cross product of any vector with itself is zero: $\vec{a} \times \vec{a} = \vec{b} \times \vec{b} = 0$
- cross product of any unit vector with itself is zero: $\hat{i} \times \hat{i} = (1)(1)\sin(0) = 0$,
 $\hat{j} \times \hat{j} = (1)(1)\cos(0) = 0$, $\hat{k} \times \hat{k} = (1)(1)\cos(0) = 0$
- any cyclic product of the three coordinate axes is **positive** and any anticyclic product is **negative** as shown bellow.



Let

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

and

$$\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

$$\vec{a} \times \vec{b} = (a_x \hat{i} + a_y \hat{j} + a_z \hat{k}) \times (b_x \hat{i} + b_y \hat{j} + b_z \hat{k})$$

$$\vec{a} \times \vec{b} = a_x \hat{i} \times b_x \hat{i} + a_x \hat{i} \times b_y \hat{j} + a_x \hat{i} \times b_z \hat{k} + a_y \hat{j} \times b_x \hat{i} + a_y \hat{j} \times b_y \hat{j} + a_y \hat{j} \times b_z \hat{k} + a_z \hat{k} \times b_x \hat{i} + a_z \hat{k} \times b_y \hat{j} + a_z \hat{k} \times b_z \hat{k}$$

$$\vec{a} \times \vec{b} = 0 + (a_x b_y) \hat{k} - (a_x b_z) \hat{j} - (a_y b_x) \hat{k} + 0 + (a_y b_z) \hat{i} + (a_z b_x) \hat{j} - (a_z b_y) \hat{i} + 0$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y) \hat{i} + (a_z b_x - a_x b_z) \hat{j} + (a_x b_y - a_y b_x) \hat{k}$$

Or using determinat form

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{i} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} \hat{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{k} = (a_y b_z - a_z b_y) \hat{i} + (a_z b_x - a_x b_z) \hat{j} + (a_x b_y - a_y b_x) \hat{k}$$

Class work

- Which of the following statements is true about the relation-ship between the dot product of two vectors and the product of the magnitudes of the vectors? (a) $\vec{A} \cdot \vec{B}$ is larger than AB; (b) $\vec{A} \cdot \vec{B}$ is smaller than AB; (c) $\vec{A} \cdot \vec{B}$ could be larger or smaller than AB, depending on the angle between the vectors; (d) $\vec{A} \cdot \vec{B}$ could be equal to AB.
- Which of the following is equivalent to the following scalar product: $(\vec{A} \times \vec{B}) \cdot (\vec{B} \times \vec{A})$? (a) $\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A}$ (b) $(\vec{A} \times \vec{A}) \cdot (\vec{B} \times \vec{B})$ (c) $(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B})$ (d) $-(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B})$

3. Which of the following statements is true about the relationship between the magnitude of the cross product of two vectors and the product of the magnitudes of the vectors? (a) $|\vec{A} \times \vec{B}|$ is larger than AB ; (b) $|\vec{A} \times \vec{B}|$ is smaller than AB ; (c) $|\vec{A} \times \vec{B}|$ could be larger or smaller than AB , depending on the angle between the vectors; (d) $|\vec{A} \times \vec{B}|$ could be equal to AB .
4. Is the triple product defined by $A \cdot (B \times C)$ a scalar or a vector quantity? Explain why the operation $A \cdot (B \times C)$ has no meaning.
5. Vector A is in the negative y direction, and vector B is in the negative x direction. What are the directions of (a) $A \times B$ (b) $B \times A$?
6. Given $\mathbf{M} = 6\hat{i} + 2\hat{j} - \hat{k}$ and $\mathbf{N} = 2\hat{i} - \hat{j} - 3\hat{k}$ calculate the $\mathbf{M} \cdot \mathbf{N}$, $\mathbf{M} \times \mathbf{N}$ and the angle between $\mathbf{M}$ & $\mathbf{N}$ and.

Short reference material

Short reference material

Chapter 2

Kinematics

Learning competencies

- Describe Kinematical terms such as distance, Displacement, average speed (velocities) and instantaneous speed (velocity).
- Solve numerical problems involving average velocity and instantaneous velocity.
- Derive equations of motion for uniformly accelerated motion.
- Apply equations of uniformly accelerated motion in solving problems.
- Relate scientific concepts to issues in everyday life.
- Explain the science of kinematics underlying familiar facts, observations and related phenomena.
- Describe the conditions at which freely falling bodies attain their terminal velocity.
- Define, Analyse and predict, terms in 2D motion
- Apply equations to solve problems related 2D motion.
- Distinguish between uniform and non-uniform circular motion.
- Analyse and predict, in quantitative terms, and explain uniform circular motion in the horizontal and vertical planes with reference to the forces involved.

2.1 Kinematics of the particle

The word Kinematics comes from Greek word "kinesis" meaning motion, thus

Kinematics: is a branch of mechanics that describes the motion of an object without refer-

ence to cause of motion (force). It does not give any information about force that causes it to move.

2.1.1 One or two dimensional (2D) motion

What is motion?

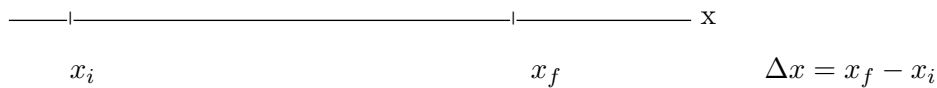
Motion is continuous change of position with time. Position is location of an object with respect to a chosen **reference frame or point**. Reference frame, also called frame of reference, in dynamics, system of graduated lines symbolically attached to a body that serve to describe the position of points relative to the body.

In physics we are considered three type of motion

1. **Translational motion:** is type motion in which all points (parts) of an object move the same distance in a given a given time. Example: A car moving in a straight line, a bullet which gets fired moves in rectilinear motion, child going down, a bird flying in the sky. In the above example, all the points of the body/object in motion are in the same direction. Translational motion can be of two types, rectilinear and curvilinear. Rectilinear motion is when an object in translational motion moves in a straight line motion. When an object in translational motion moves along a curved path, it is said to be in curvilinear motion
- 2 **Rotational motion:** is when an object moves about an axis and different parts of it move by different distances in a given interval of time. Examples: blades of a rotating fan, merry-go-round, blades of a windmill. When an object undergoes rotational motion, all its parts do not move the same distance in a given interval of time. For example, the outer portion of the blades of a windmill moves much more than the portion closer to the centre.
- 3 **Vibrational motion:** is when a body moves to and fro about its mean position is called vibratory motion. Vibratory motion can be described as any object moving/swinging back and forth, moving up and down, pulsating, etc. **Examples** Pendulums, swings, tuning forks, etc are of vibratory motion. Vibrational motion can be periodic or non-priodic

2.1.2 Motion in 1D

This is motion of a particle along straight line in fixed direction (or motion along one coordinate axis). Example: a car moving along a flat straight narrow road.



Definition of Kinematical terms

Distance and Displacement

Distance: is the total path length covered by the moving object.

Displacement: is change of position i.e the shortest distance between start and end of motion. For example: a particle moving from point A to B as shown in figure below.

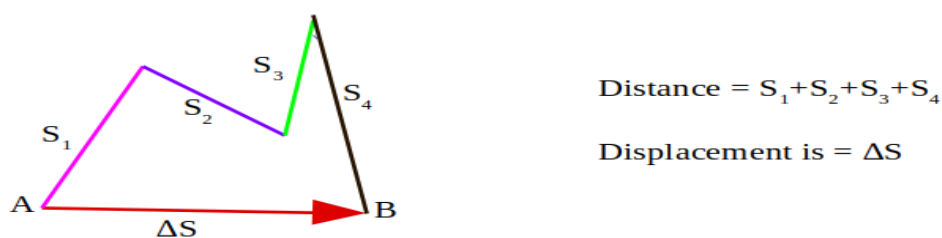


Figure 2.1: Comparison of Distance and Displacement

Speed and Velocity

Speed (v): is the rate of change of distance in a unit time.

Average Speed (v_{av}): is total distance traveled by the total time required to cover the distance.

$$v_{av} = \frac{\text{total distance}}{\text{total time}}$$

Velocity ($\vec{v}$): the rate of change of displacement as a function of time.

Average Velocity ($\vec{v}_{av}$): is change of displacement Δx divided by the time interval Δt during which the displacement occurs.

$$\vec{v}_{av} = \frac{\vec{x}_f - \vec{x}_i}{t_f - t_i} = \frac{\Delta \vec{x}}{\Delta t}$$

Instantaneous Velocity and Speed

Instantaneous Velocity $\mathbf{v(t)}$: is the Velocity of the particle at a given instant of time. It is the limit of average velocity as Δt approaches to zero.

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t + \Delta t) - \vec{x}(t)}{\Delta t}$$

This can be rewritten as first derivatives of displacement with respect to time.

$$v(t) = \frac{dx}{dt}$$

The magnitude of Instantaneous velocity is instantaneous **speed**

Class work

1. Which of the following is true for displacement?
 - (a) It cannot be zero.
 - (b) Its magnitude is greater than the distance travelled by the object.
 - (c) displacement may or may not be equal to distance
2. If the displacement of the body is zero, the distance covered by it may not be zero.
3. In which of the following cases of motions, the distance moved and the magnitude of displacement are equal ?
 - (a) If the object is moving along straight road
 - (b) If the object is moving along straight path
 - (c) The pendulum is moving back and forth
 - (d) The earth revolving around the sun
4. A particle moves along the x-axis according to the equation given below.

$$\vec{x}(t) = (4 + 2t - t^2)m\hat{i}$$

where t is in Second.

- a) Determine the displacement of this particle between the time interval $t = 0$ and $t = 1s$
 - b) Determine the average velocity during those two time intervals
 - c) Derive a general expression for the instantaneous velocity as a function of time.
 - d) calculate instantaneous velocity at $t = 2s$.
5. A boy walks from his home to school at constant speed of 5m/s along straight line and then back along the same line (road) from school to his home at constant speed of 6m/s.

- a) What is his average speed?
- b) What is his average velocity?

Acceleration ($\vec{a}$): is the rate of change of velocity.

Average acceleration ($\vec{a}_{av}$): is the change in velocity divided by time interval during which it occurs.

$$\vec{a}_{av} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$$

Instantaneous acceleration $\vec{a}(t)$: is the acceleration of the particle at a given instant of time. It is the limit of average acceleration as Δt approaches to zero.

$$\vec{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}$$

This can be rewritten as Second derivatives of displacement or first derivatives of velocity with respect time.

$$\vec{a}(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Class work

1. A particle moves along the x-axis component varies with time according to equation given below.

$$\vec{x}(t) = (20 - 2t + t^3)m\hat{i}$$

where t is in Second.

- a) Determine initial position of the particle.
- b) Determine the displacement of this particle between the time interval $t = 0$ and $t = 1s$; $t = 1$ and $t = 4s$
- c) Determine the average velocity during those two time intervals
- d) Derive a general expression for the instantaneous velocity as a function of time.
- e) calculate instantaneous velocity at $t = 3s$.
- f) average acceleration between $t = 2s$ to $t = 3s$.
- g) Derive a general expression for the instantaneous acceleration as a function of time.
- h) calculate instantaneous acceleration at $t = 2s$.

Uniform Motion in 1D

Uniform Motion: A body is said to be in a state of uniform motion if it travels equal distances in equal intervals of time. If the time distance graph is a straight line the motion is said to be uniform motion. This means that the velocity of the body remains constant as it covers equal distance in equal interval of time, in case of uniform rectilinear motion acceleration of the body will be zero. Here, the average speed and instantaneous speed will be equal to the actual speed; average velocity and instantaneous velocity will be equal to the actual velocity and the magnitude of velocity is equal speed.

$$\vec{v}_{av} = \vec{v} = \vec{v}(t)$$

$$|\vec{v}| = v$$

$$\Delta x = \vec{v}(t)t$$

Uniformly accelerated Motion in 1D

This is motion with constant acceleration i.e. velocity change with uniform rate.

$$\vec{a}(t) = \frac{dv}{dt} = \frac{v - v_i}{t} = \text{constant}$$

$$v(t) = v_i + at \quad (2.1)$$

Average velocity for uniformly accelerated motion is given by

$$\vec{v}_{av} = \frac{\vec{v} + \vec{v}_i}{2}$$

Thus

$$\vec{x} - \vec{x}_i = \Delta \vec{x}t = \vec{v}_{av}t = \left(\frac{\vec{v} + \vec{v}_i}{2}\right)t \quad (2.2)$$

Using equation (2.1) into (2.2)

$$\begin{aligned} \Delta \vec{x} &= \left(\frac{\vec{v}_i + at + \vec{v}_i}{2}\right)t \\ \Delta \vec{x} &= v_it + \frac{1}{2}at^2 \end{aligned} \quad (2.3)$$

From equation (2.1)

$$t = \frac{v - v_i}{a} \quad (2.4)$$

Using equation (2.4) into (2.2)

$$\Delta x = \left(\frac{v + v_i}{2}\right)\left(\frac{v - v_i}{a}\right) = \frac{v^2 - v_i^2}{2a}$$
$$v^2 = v_i^2 + 2a\Delta x \quad (2.5)$$

Class work

1. An electron in a cathod ray tube accelerate uniformly from $2.0 \times 10^4 m/s$ to $6.0 \times 10^6 m/s$ over 1.5cm.
 - a) How long does the electron take the to travel this 1.5cm?
 - b) What is its acceleration?
2. A track covers 40m in 8.5s while smoothly slowing down to a final speed of 2.8m/s. Find
 - a) its original velocity
 - b) its acceleration
3. A jet lands on an air craft at 140mi/hr and stops in 2s due to an arresting cable that snags the air plane.
 - a) What is its acceleration?
 - b) If the plane touches down at position $x_i = 0$ what is the final position of the plane?
4. A car traveling at constant speed of 45m/s passes a tropper hidden behind a billboard. One second after the speeding car passes the billboard, the tropper sets out from the billboard to catch it, accelerating at constant rate of $3.0m/s^2$. How long does it take her to over take the car?
5. A jet plane lands with a speed of 100m/s and slow down at rate of $5m/s^2$ as it comes to rest.
 - a) What is the time interval needed by the jet to come to rest?
 - b) Can this jet land on an airport where the runway is 0.8km long?

Free falling bodies

A freely falling object is any object moving freely under the influence of gravity alone, regardless of its initial motion.

Example: object thrown upward or down ward and object released from rest.

Free fall is motion with constant gravitational acceleration $g = 9.81m/s^2$ toward the center of the earth. So we can use equation of uniformly accelerated as in table below

When released from rest When thrown up When thrown down

$v_y = gt$	$v_y = v_{oy} - gt$	$v_y = v_{oy} + gt$
$\Delta y = \frac{1}{2}gt^2$	$\Delta y = v_{oy}t - \frac{1}{2}gt^2$	$\Delta y = v_{oy}t + \frac{1}{2}gt^2$
$v_y^2 = 2g\Delta y$	$v_y^2 = v_{oy}^2 - 2g\Delta y$	$v_y^2 = v_{oy}^2 + 2g\Delta y$

Class work

1. A girl thows a ball upwards, moving it an initial speed $u = 15m/s$. Neglect air resistance
 - a) How long does the ball take to return to the girl's hand?
 - b) What will be its velocity then?
2. A ball is thrown upward. While the ball is in free fall, does its acceleration (a) increase (b) decrease (c) increase and then decrease (d) decrease and then increase (e) remain constant?
3. After a ball is thrown upward and is in the air, its speed (a) increases (b) decreases (c) increases and then decreases (d) decreases and then increases (e) remains the same.

2.1.3 Two dimension (2D) motion

2D motion is motion in a plane (This means object moving along two coordinate axis simultaneously, and its position can be described by two coordinate). Example: Projectile motion, Circular motion

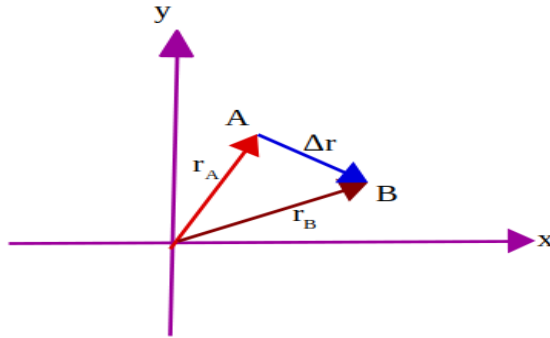


Figure 2.2: Motion in a plane

If a particle move from point A to point B in figure 2.2 its displacement is given by

$$\Delta \vec{r} = \vec{r}_B - \vec{r}_A$$

$$\Delta \vec{r} = (x_B \hat{i} + y_B \hat{j}) - (x_A \hat{i} + y_A \hat{j}) = \Delta x \hat{i} + \Delta y \hat{j}$$

For infinitesimal change

$$d\vec{r} = dx \hat{i} + dy \hat{j}$$

Average velocity (v_{av}) is given by

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t} \hat{i} + \frac{\Delta y}{\Delta t} \hat{j} = \vec{v}_x \hat{i} + \vec{v}_y \hat{j}$$

Instantaneous velocity is given by

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

$$v(t) = \vec{v}_x(t) \hat{i} + \vec{v}_y(t) \hat{j}$$

Average acceleration is given by

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{\Delta v_x}{\Delta t} \hat{i} + \frac{\Delta v_y}{\Delta t} \hat{j}$$

Instantaneous acceleration

$$a(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}$$

$$a(t) = \vec{a}_x(t)\hat{i} + \vec{a}_y(t)\hat{j}$$

Class work

1. A bird flies in xy plane with velocity vector given by

$$\vec{v} = (\alpha - \beta t^2)\hat{i} + \gamma t\hat{j}$$

where $\alpha = 2.1\text{m/s}$ and $\gamma = 2.8\text{m/s}^2$ and the positive y direction is vertically upward at $t = 0$, the bird is at the origin.

- a) Determine average acceleration between time interval $t = 0$ to 1s
- b) calculate the general expression for instantaneous acceleration at any time t
- c) What is the birds altitude (y coordinate) as it flies over $x = 0$ for the first time after $t = 0$.

2.1.3.1 Projectile Motion

Projectile motion is motion of an object in a plan under the influence of gravity alone, regardless of its initial motion (neglecting air resistance). Examples: A ball kicked from the horizontal ground. The path followed by projectile motion is trajectory and downward

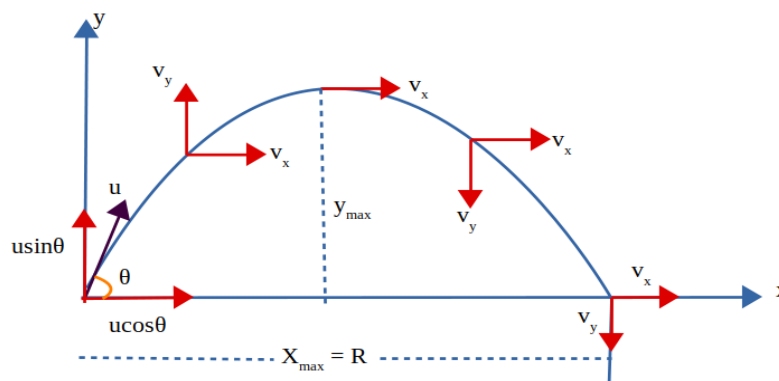


Figure 2.3: *Projectil Motion*

parabola due to gravitational acceleration and combination of horizontal and vertical velocity as we can see in figure 2.3. As projectile motion is 2D motion we can regard it as two separate and independent horizontal (x-component) and vertical (y-component) motion.

Horizontal motion of projectile

Horizontal motion of projectile motion is uniform motion (velocity constant, $a_x = 0$). Because no net force act on horizontal motion of projectile motion. ie

$$v_{0x} = v_0 \cos \theta$$

this is horizontal component of initial velocity.

$$v_x = v_{0x} = v_0 \cos \theta = \text{constant}$$

and

$$x(t) = v_{0x}t = v_x t = v_0 \cos \theta t$$

Vertical motion of projectile

Vertical motion of projectile motion is uniformly accelerated motion. It is motion with constant gravitational acceleration of $g = 9.8 \text{ m/s}^2$ to ward the center of the earth. At the origin

$$v_{0y} = v_0 \sin \theta$$

this is vertical component of initial velocity.

Now we can use equation uniformly accelerated motion as

$$v_y = v_{0y} - gt$$

$$\Delta y = \frac{(v_{0y} + v_y)}{2} t$$

$$\Delta y = v_{0y} t - \frac{1}{2} g t^2$$

$$v_y^2 = v_{0y}^2 - 2g\Delta y$$

But at the maximum height $v_y = 0$ so

$$0 = v_{0y}^2 - 2gy_{max}$$

$$y_{max} = \frac{v_{0y}^2}{2g} = \frac{v_0^2 \sin^2 \theta}{2g}$$

Total time (t) is time of flight is given by

$$v_y = v_{0y} - gt$$

but at $y_{max}, v_y = 0$

$$0 = v_{0y} - gt_a$$

$$t_a = \frac{v_{0y}}{g} = \frac{v_0 \sin \theta}{g}$$

But total time is $t = t_a + t_d$ and $t_a = t_d$ thus

$$t = \frac{2v_{0y}}{g} = \frac{2v_0 \sin \theta}{g}$$

Range (R) is maximum horizontal displacement (x_{max})

$$R = v_{0x}t = v_0 \cos \theta \left(\frac{2v_0 \sin \theta}{g} \right)$$

$$R = \frac{v_0^2 2 \cos \theta \sin \theta}{g}$$

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

The maximum range is reached at an angle of projection $\theta = 45^\circ$

$$R = \frac{v_0^2 \sin(2 \times 45^\circ)}{g} = \frac{v_0^2 \sin(90^\circ)}{g} = \frac{v_0^2}{g}$$

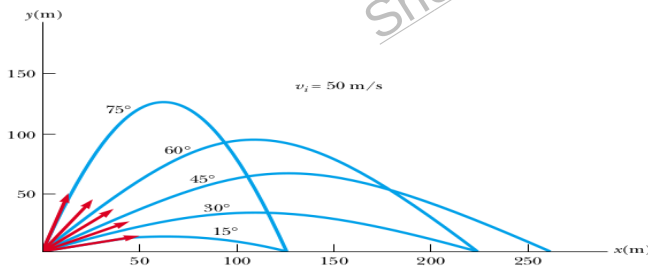


Figure 2.4: A projectile launched from the origin with an initial speed of 50 m/s at various angles of projection. Note that complementary values of θ_i result in the same value of R (range of the projectile).

Class work

1. A ball is kicked with an initial velocity of 40m/s from the ground at an angle of 30° to the horizontal. (Use $g = 10\text{m/s}^2$) Calculate
 - a) Horizontal and vertical component of initial velocity
 - b) The vertical velocity after $t = 1\text{s}, 2\text{s}, 3\text{s},$ and 4s

- c) Position ($\vec{r} = x\hat{i} + y\hat{j}$) after $t = 1\text{s}, 2\text{s}, 3\text{s},$ and 4s
 - d) Time of flight (total time)
 - e) maximum height
 - f) Range of projectile
2. An air plane moving horizontally with velocity of 500km/hr at a height of 2km above the ground dropped a bomb when it directly above the target. By how much distance will the bomb miss the target?
3. An astronaut on a strange planet finds that she can jump a maximum horizontal distance of 15.0m if her initial speed is 3.00m/s . What is the free-fall acceleration on the planet?

Short reference material

2.1.3.2 Circular Motion

Uniform Circular Motion

Uniform circular motion: is a type of motion in which an object moves in a circular path at a constant speed. The direction of motion is constantly changing as the object moves around the circle. For example, imagine a car moving around a circular racetrack at a constant speed of 100 km/h. The car is always moving in a circle, and the direction of the car is constantly changing as it goes around the track. However, the car's speed is always the same, so the car is said to be in uniform circular motion. Another example of uniform circular motion is a planet orbiting around a star. The planet is constantly moving in a circular path around the star, and its speed is constant as it moves around the orbit.

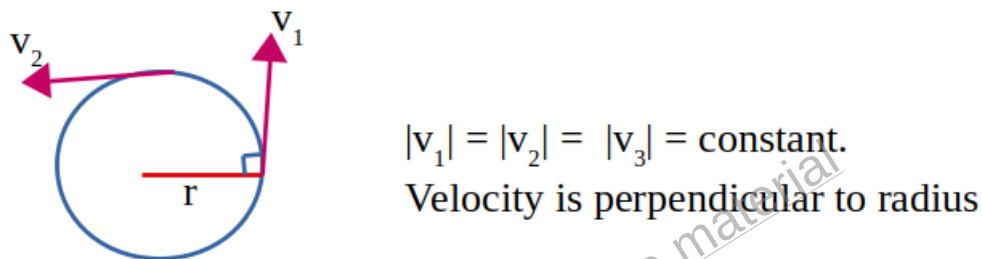
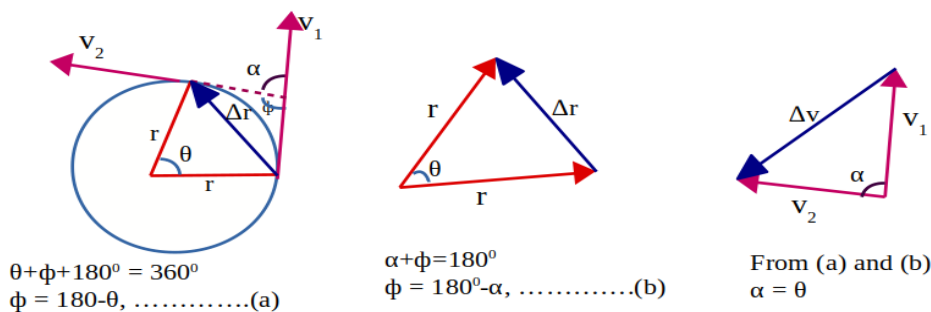


Figure 2.5: *Uniform Circular Motion.*

In uniform circular motion velocity is not constant because continuous variation of direction.



So the two triangle are similar by $\hat{S}\hat{A}\hat{S}$. For similar triangle the ratio of their side is equal

$$\frac{|\Delta \vec{r}|}{|\Delta \vec{v}|} = \frac{|\vec{r}|}{|\vec{v}|} \text{ or } \frac{|\Delta \vec{v}|}{|\Delta \vec{r}|} = \frac{|\vec{v}|}{|\vec{r}|}$$

$$\Delta v = \frac{v}{r} \times \Delta r$$

Dividing bothside by Δt

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta r}{\Delta t}$$

$$a_c = \frac{v}{r} \times v$$

$$a_c = \frac{v^2}{r}$$

Thus a_c is **radial or centripetal** acceleration and it is always toward the center of the circle perpendicular to velocity. Its magnitude is constant but its direction change continuous perpendicular to velocity. This acceleration is due to a **centripetal force**. A centripetal force is a net force that acts on an object to keep it moving along a circular path and its direction is toward the center the circle. **Example:** The tension force in the string of a swinging tethered ball and the gravitational force keeping a satellite in orbit are both examples of centripetal forces. Multiple individual forces can even be involved as long as they add up (by vector addition) to give a net force towards the center of the circular path.

Period (T): it is time taken for one complete rotation.

$$T = \frac{2\pi r}{v}$$

Non-uniform Circular Motion

Non-uniform circular motion is a type of circular motion in which the speed of an object moving in a circular path changes at different points along the path. In other words, the magnitude of the velocity vector of the object is not constant, meaning that the object is accelerating even though it is moving in a circle.

An example of non-uniform circular motion is a car driving around a curved road. The car's speed may change as it navigates the curve, depending on factors such as the car's position on the curve and the road conditions. The car's direction is constantly changing as it moves around the curve, but the speed is not constant. As a result, the car is undergoing non-uniform circular motion. Another example of non-uniform circular motion is a planet orbiting a star, as the planet speeds up and slows down in its elliptical orbit. In this case there are two type of acceleration:-

1. **Radial or centripetal acceleration:-** due to change of direction of motion

$$\vec{a}_c = \frac{v^2}{r}$$

2. **Tangential acceleration:-** due to change magnitude of velocity. Its magintude is

change of velocity over change of time.

$$\vec{a}_T = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\Delta v}{\Delta t}$$

Its direction is in the direction of Velocity which is perpendicular to centripetal acceleration. Thus total acceleration

$$\vec{a} = \vec{a}_c + \vec{a}_T$$

Its magnitude

$$a = \sqrt{(\vec{a}_c)^2 + (\vec{a}_T)^2}$$

Its direction

$$\theta = \tan^{-1} \left(\frac{a_T}{a_c} \right)$$

Class work

1. A ball tied to the end of a string 1m in length swings in a vertical circle under the influence of gravity. When the string makes an angle of 20° its speed was 2m/s. Calculate
 - a) magnitude of centripetal acceleration
 - b) magnitude of centripetal Tangential acceleration
 - c) magnitude and direction of total acceleration

Chapter 3

Angular Motion

Learning competencies

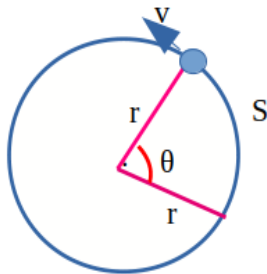
- Describe the rotational kinematical quantities.
- Give the angular speed and angular velocity of a rotating body.
- Determine the velocity of a point in a rotating body.
- Derive equations of motion with constant angular acceleration.
- Use equations of motion with constant angular acceleration to solve related problems.
- State the law of conservation of angular momentum.
- Apply the law of conservation of angular momentum in Understanding various natural phenomena, and solving problems.
- Express angular momentum as a cross product of $\mathbf{r}$ and $\mathbf{p}$.
- Derive an expression for angular momentum in terms of I and ω .
- Use the relationship between torque and angular momentum, according to Newton's second law.
- Apply the relationship between torque and angular momentum to solve problems involving rigid bodies.

3.1 Angular Motion

Angular motion is a type of motion that occurs when an object moves along a circular path or rotates around a fixed axis. Angular motion is characterized by two main quantities: angular displacement and angular velocity. Angular motion is important in many areas of physics, including mechanics, electromagnetism, and quantum mechanics. It is also used in many practical applications, such as in the design of engines, turbines, and other rotating machinery.

3.1.1 Rotational Kinematics

Angular displacement: is the change in the angle (θ) of rotation of an object with respect to a fixed axis as we can see in figure below. Radian (rad) is SI unit of angular displacement,



Angular displacement (θ) is measured in radians (or degrees or revolution) and is equal to the ratio of the length of the arc swept out by the object to the radius of the circular path.

$$\theta = S/r$$

one radian is angle subtended by an arc length equal to radius of the arc. The relation between revolution (rev), degree (deg or $^\circ$) and radian (rad)

$$2\pi \text{ rad} = 360^\circ = 1 \text{ rev} \quad (3.1)$$

Average Angular velocity: is the rate of change of the angular displacement of an object with respect to time. It is represented by Greek letter ω . It is measured in radians per second or (degrees per second or revolution per second) and is equal to the ratio of the change in the angular displacement of the object to the time interval over which the change occurred.

$$\vec{\omega} = \frac{\theta - \theta_0}{t - t_0} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Velocity: It is average angular velocity as $\Delta t \rightarrow 0$. This means angular velocity at instant of time (for infinitesimal change)

$$\omega(t) = \frac{d\theta(t)}{dt}$$

Angular acceleration: the rate of change of the angular velocity of an object with respect to time. It is denoted by Greek letter α .

Average angular acceleration (ω_{av}): is the ratio of angular velocity to time interval Δt during which the change occurs.

$$\vec{\alpha} = \frac{\vec{\omega} - \vec{\omega}_0}{t - t_0} = \frac{\Delta\vec{\omega}}{\Delta t}$$

Instantaneous Angular acceleration: It is average angular acceleration as $\Delta t \rightarrow 0$. This means angular acceleration at instant of time (for infinitesimal change). Its SI unit is rad/s^2

$$\alpha(t) = \frac{d\omega(t)}{dt} = \frac{d^2\theta(t)}{dt^2}$$

Uniformly Accelerated Rotational Motion

Uniformly accelerated angular motion refers to the motion of an object rotating around an axis at a constant rate of acceleration (α). This means that the angular velocity of the object is changing at a constant rate over time. In this type of motion, the angular acceleration of the object is constant, which means that the rate of change of the angular velocity is also constant. The equation that describes this relationship is:

$$\omega = \omega_0 + \alpha t \quad (3.2)$$

For uniformly accelerated angular motion the average angular velocity is given by

$$\omega_{av} = \frac{\omega + \omega_0}{2}$$

Therefore

$$\theta - \theta_0 = \left(\frac{\omega + \omega_0}{2}\right)t \quad (3.3)$$

Using all those together

$$\theta = \omega_0 t + \frac{1}{2}\alpha^2 t^2 \quad (3.4)$$

and

$$\omega^2 = \omega_0^2 + 2\alpha\theta \quad (3.5)$$

Class work

1. The angular position of a fly wheel of car engines is given by

$$\theta = (2\text{rad/s}^3)t^3$$

the diameter of the flywheel is 0.36m.

- a) Find the angle θ in radian, degree and revolution at time $t_1 = 2\text{sec}$ and $t_2 = 5\text{sec}$.
- b) Find the distance that the particle of the rim moves during that time interval.
- c) Find the average angular velocity in rad/s , and rev./m

- d) Find the general expression for the angular velocity at any time t .
- e) Find the general expression for the angular acceleration at any time t .
2. A wheel rotates with angular acceleration of 3.5 rad/s^2 . If the angular speed of the wheel is 2.0 m/s at $t=0$.
- a) through what angle does the wheel rotate in 2.0 s
- b) what is the angular speed at $t = 2.0 \text{ s}$

Angular Momentum

The angular momentum ($\vec{L}$) of a moving particle with respect to a given axis is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

Where $\vec{r}$ is distance from axis of rotation and $\vec{p} = m\vec{v}$ is linear momentum.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

and we know that $v = r\omega$

$$\vec{L} = m\vec{r} \times r\vec{\omega} = mr^2\vec{\omega}$$

And we know that $I = mr^2$

$$\vec{L} = I\vec{\omega} \quad (3.6)$$

Eqn. 3.6 is angular momentum. Taking time derivatives of equation 3.6

$$\frac{d\vec{L}}{dt} = m\vec{r} \times \frac{d\vec{v}}{dt} = m\vec{r} \times \vec{a} = \vec{r} \times \vec{F}_{net} = \vec{\tau}_{net}$$

Law of Conservation of Angular Momentum

This states that if the net external torque acting on the system is zero (the system is isolated) then the angular momentum of the system is conserved (remain unchanged)

$$\vec{L}_i = \vec{L}_f$$

$$I_i\omega_i = I_f\omega_f \quad (3.7)$$

Class work

1. The position vector of a particle of mass 2kg is given as a function of time by $\vec{r} = (6\hat{i} + 5t\hat{j})$. Determine the angular momentum of the particle about the origin as a function of time.
2. A large circular disk of mass 2kg and radius 0.2m initially rotating at 50rad/s is coupled a smaller circular disk of mass 4kg and radius 0.1m initially rotating at 20rad/s in the same direction as large disk.
 - a) Find the common angular velocity after the disk are coupled.
 - b) Calculate the loss of kinetic energy during this collision.

Short reference material

Short reference material

Chapter 4

Dynamics

Learning competencies

- Identify the four basic forces in nature.
- Define and describe the concepts and units related to force.
- Define the term dynamics.
- Define and describe the concepts and units related to coefficients of friction.
- Use the laws of dynamics in solving problems.
- Interpret Newton's laws and apply these to moving objects.
- Explain the conditions associated with the movement of objects at constant velocity.
- Solve dynamics problems involving friction.
- Analyse, in qualitative and quantitative terms, the various forces acting on an object in a variety of situations, and describe the resulting motion of the object.
- Describe the terms momentum and impulse.
- State the law of conservation of linear momentum.
- Discover the relationship between impulse and momentum, according to Newton's second law.
- Apply quantitatively the law of conservation of linear momentum.
- Distinguish between elastic and inelastic collisions.
- Describe head-on collisions.
- Describe glancing collisions.
- Define and describe the concepts and units related to torque.
- Describe centre of mass of a body.
- Determine the position of centre of mass of a body.
- Interpret Newton's laws and apply these to objects undergoing uniform circular motion.
- Solve dynamics problems involving friction.

4.1 Dynamics

Dynamics: In physics, dynamics is the branch of mechanics that deals with the study of motion and the forces that cause or affect that motion. It involves the analysis of how an object moves and the forces that cause it to move, including the study of the forces that cause changes in the motion of an object, such as acceleration, deceleration, and changes in direction. The fundamental concepts in dynamics are force, mass, and acceleration, as described by Newton's laws of motion. Dynamics is used to describe a wide range of physical phenomena, from the motion of particles at the subatomic level to the motion of planets in the solar system. It is used in many fields, including engineering, physics, and applied mathematics, to understand and predict the behavior of physical systems.

Force is a physical quantity that describes an interaction between two objects that can cause a change in motion of one or both objects. It is defined as the product of mass and acceleration, or more formally as the rate of change of momentum with respect to time. It is a push or a pull of an object (Interaction that change state of motion). We can't see force with our naked-eye but, in everyday life, we experience the following effects of force all the time.

- Force set or tends to set an object to motion
- Force stop or tends to stop motion
- Force change direction of motion
- Force accelerate or decelerate motion
- Force change shape and size of materials

Type of force

Force usually categorized into two

1. **Contact Force:** This is a force that requires physical contact between two objects in order for the force to be applied. **Examples:** Frictional force, Tension force, Normal force, Air resistance force, and Applied force.
2. **Non-contact Force:** This is a force that can act over a distance without any physical contact between the objects. **Examples:** Gravitational force, Magnetic force, Electrostatic force, Electromagnetic force, Nuclear force.

Newton's Law of Motion

Newton's laws of motion are three fundamental principles that describe the behavior of objects in motion. They were first introduced by Sir Isaac Newton in his 1687 work "Philosophiæ Naturalis Principia Mathematica". The laws are:, 1. Newton's first Law of motion (Law of inertia), 2. Newton's second law (Law of acceleration) and 3. Newton's third law (action and reaction force)

Newton's first law: This states that "Unless an external force exerted on the body the state of motion the body remain as it is". This is called law of inertia. **Inertia:** is the tendency of the body to resist its change of state of motion.

Newton's second law: This states that "acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass".

$$\sum \vec{F} = m\vec{a}$$

$$\sum \vec{F} = \sum \vec{F}_x + \sum \vec{F}_y + \sum \vec{F}_z = m(\vec{a}_x + \vec{a}_y + \vec{a}_z)$$

Newton's third law: This states that "if object A exert force on object B, then object B exert a force on object A that is equal in magnitude and opposite in direction". Thus force always occur in pair. This pair of force are called action and reaction force. For every action force there is reaction force. Action and reaction force are always:

- the same in magnitude
- opposite in direction
- act on different bodies
- the same type

Class work

1. Object of mass 10kg is exerted on by a force of $\vec{F}_1 = (2\hat{i} + 3\hat{j})N$, $\vec{F}_2 = (4\hat{i} - 3\hat{j})N$ and $\vec{F}_3 = (-\hat{i} + 3\hat{j})N$, calculate
 - a) net force on this object
 - b) its acceleration
2. A 2kg object undergoes an acceleration of given by $\vec{a} = (3\hat{i} + 4\hat{j})m/s^2$. Find the magnitude resultant of force.

3. A particle of mass 2unit moves along space curve defined by $\vec{r}(t) = (4t^2 - t^3)\hat{i} - 5t\hat{j} + (t^4 - 2)\hat{k}$. Find the force acting on it at any time t.
4. Find the force needed to accelerate a mass of 400kg from velocity $\vec{v}_0 = (4\hat{i} - 5\hat{j} + 3\hat{k})m/s$ to $\vec{v}_f = (8\hat{i} + 3\hat{j} - 5\hat{k})m/s$ in 10s.

Friction Force

Friction force: is a force generated in opposite direction to the motion when solid object slide or attempt to slid over each other. Its magnitude is given by

$$f = F_N \mu$$

Where μ is coefficient of friction (constant that depend on the nature of the surface in contact), F_N is normal force. There are two type of frictional force

- **Static friction:-** friction occure when object attempt to slid over each other but not yet slid over each other. Its magnitude given

$$f_s = F_N \mu_s$$

μ_s is coefficient of static friction

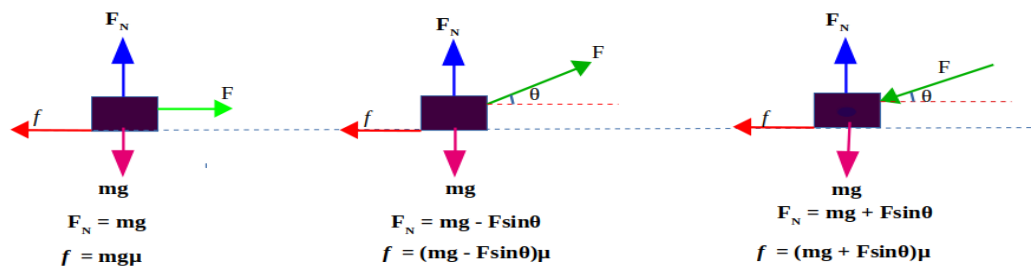
Kinetic friction:- friction force occure when object sliding over eachother

$$f_s = F_N \mu_k$$

μ_k is coefficient of kinetic friction

NB $\mu_s > \mu_k$ thus $f_s > f_k$

Normal Force (F_N):- Is a force or component of force that is perpendicular to the surface in contact and equal in magnitude to the force that holds the surface press together. Normal force equal to mg when the sliding object is on horizontal surface and acted on by horizontal force as shown in figure below. But if the force acted on the object is at a certain angle to the horizontal the normal force is different as shown in figure below



Class work

1. A 20kg block is initially at rest on a horizontal surface. A horizontal force of 75N is a required to set the block in motion. After it is in motion a horizontal force of 60N is required to keep the block moving with constant speed. Find the coefficient of static and kinetic friction.

Applied Force: Applied force is a physical force that is applied to an object by a person or another object. It is a force that causes an object to move, accelerate, or change direction. Applied force is an important concept in physics and is used to describe many physical phenomena, including the motion of objects, the behavior of fluids, and the behavior of electromagnetic fields.

Gravitational force is the force by which a planet or other body draws objects toward its center. The force is always attractive and acts along the line connecting the two bodies. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between them. The proportionality constant is known as the gravitational constant. The gravitational force is responsible for keeping the planets in orbit around the sun and for keeping the moon in orbit around the Earth. The mathematical formula for the gravitational force between two objects can be expressed using Newton's law of gravitation:

$$F = G \frac{m_1 m_2}{r^2}$$

where:

- F is the gravitational force between the two objects, measured in Newtons (N), - G is the gravitational constant, which has a value of approximately $6.674 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$, - m_1 and m_2 are the masses of the two objects in kilograms (kg), - r is the distance between the centers of mass of the two objects, measured in meters (m)

A restoring force is a force that acts to bring an object back to its original position after it has been displaced. In other words, it is a force that opposes displacement. For example,

the force exerted by a spring when it is stretched or compressed is a restoring force. In a simple harmonic motion, restoring force is directly proportional to the displacement and acts in the direction opposite to the displacement. The mathematical expression of the restoring force for a spring is:

$$F = -kx$$

where F is the restoring force, x is the displacement from the equilibrium position, and k is the spring constant, which is a measure of the stiffness of the spring.

Another example of a restoring force is the force of gravity acting on a pendulum. The restoring force of a pendulum is given by:

$$F = -mg\sin(\theta)$$

where F is the restoring force, m is the mass of the pendulum, g is the acceleration due to gravity, and θ is the angle between the pendulum and the vertical. In general, the mathematical expression of the restoring force depends on the specific physical system being considered, and can be derived from the laws of physics governing that system.

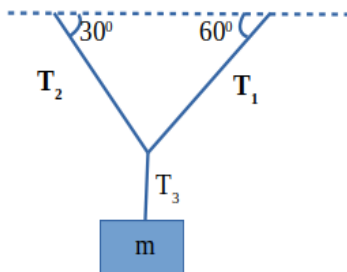
Application and Newton's law of motion

In this case we apply Newton's law to objects either in equilibrium ($\vec{a} = 0$) or accelerating along straight line under action of constant force. The following procedure is recommended when dealing problems with involving Newton's law.

1. Identify the object or particle on which force are exerted.
2. Identify the force exerted on the object (Draw free body diagram)
3. Decompose each force into their x,y and z-components.
4. Calculate net force, acceleration, velocity and so'on.

Class work

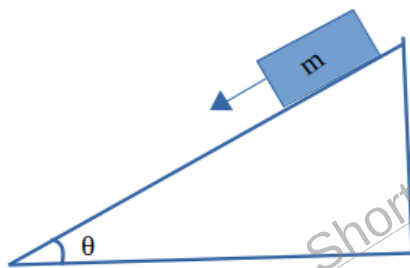
1. A block of mass 10kg hangs from three cords as shown below



What are the tension in the cords?
(Assume the cords has negligible mass)

Figure 4.1: 10kg hangs from three cords

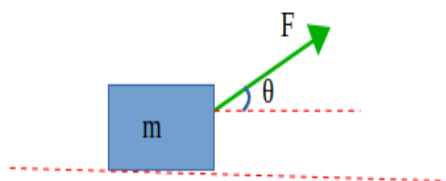
2. A block of mass m slides down an inclined plane as shown in the figure below.



Find the expression for the acceleration of the block if the inclined plane
a) is friction less
b) has coefficient of friction μ_k

Figure 4.2: mass m slides down an inclined plane

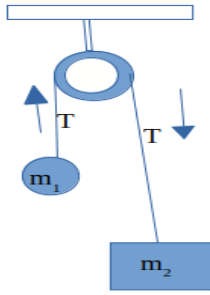
3. The block of mass m sliding horizontally as shown in figure below.



Find the expression for the acceleration of the block, if the coefficient of friction between the block and the surface is μ_k

Figure 4.3: block of mass m sliding horizontally

4. Two object of unequal mass are hung vertically over a frictionless pully of negligible mass as in figure below



Determine the expression for magnitude of acceleration of the two object and tension in the cord

Figure 4.4: unequal mass are hung vertically over a frictionless pully

4.2 linear momentum

Linear momentum ($\vec{p}$) is defined as quality of an object to exert a force on any thing that tries to change its state of motion. Linear momentum is an important concept in physics because it is a measure of an object's ability to cause change through its motion. For example, a moving car has a lot of linear momentum and is able to do a lot of damage in a collision because it is difficult to stop. On the other hand, a stationary car has no linear momentum and is not able to cause much change through its motion. Its magnitude is the product of mass of the system with its velocity.

$$\vec{p} = m\vec{v} \quad (4.1)$$

For an objet's in three dimension

$$\vec{p} = m(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}) \quad (4.2)$$

Linear momentum is a vector quantity, meaning it has both magnitude and direction. The direction of an object's linear momentum is the same as the direction of its velocity. Its SI unit is kgm/s.

Impulse ($\vec{J}$): Impulse is defined as the product of the force acting on an object and the time for which the force acts. Mathematically, impulse can be expressed as: **From Newton's second law**

$$\vec{F}_{net} = m \frac{\Delta v}{\Delta t} = \frac{\Delta}{\Delta t}(m\vec{v}) = \frac{\Delta \vec{p}}{\Delta t} \quad (4.3)$$

This can be rewritten as

$$\Delta \vec{p} = \vec{F}_{net} \Delta t = \vec{J} \quad (4.4)$$

Or

$$\vec{F}_{net} = \frac{\Delta \vec{p}}{\Delta t} \quad (4.5)$$

This is the relation between $\vec{p}$ and resultant force acting on it.

4.2.1 Conservation of Momentum

When ever two or more particles in an isolated system (in which net external force acting on the system is zero) interact, the total momentum of the system remain constant (conserved) i.e

$$\sum \vec{p}_i = \sum \vec{p}_f \quad (4.6)$$

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2 \quad (4.7)$$

Where u_1 & u_2 are initial velocity of m_1 and m_2 respectively, and v_1 & v_2 are final velocity of m_1 and m_2 respectively Thus

$$\vec{p}_{xi} = \vec{p}_{xf}, \vec{p}_{yi} = \vec{p}_{yf}, \vec{p}_{zi} = \vec{p}_{zf}$$

Class work

1. A 60 kg archer stands at rest on a frictionless ice and fires a 0.5 kg arrow horizontally at 50m/s. With what velocity does archer move across the ice after firing the arrow?

4.2.2 Collision

Collision: is the event of two particles coming together for short time and thereby producing impulsive force on each other. Collisions are an important topic in physics because they can be used to understand a wide range of phenomena, from the behavior of subatomic particles to the motion of celestial bodies in the universe. Depending on kinetic energy, the Q-value and coefficient of restitution Collision grouped in two (1) Elastic Collision, and (2) Inelastic Collision.

Elastic Collision

It is type of collision in which both kinetic energy and momentum are conserved.

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad (4.8)$$

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad (4.9)$$

The collision in which kinetic energy is fully conserved is called **perfectly elastic collision**.

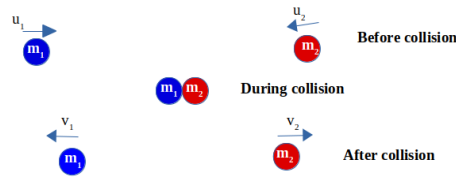


Figure 4.5: *perfectly elastic collision*

Inelastic Collision

It is a type of collision in which only momentum is conserved but kinetic energy is not conserved.

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad (4.10)$$

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \neq \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad (4.11)$$

A collision in which a colliding object stick together after collision is called **perfectly inelastic collision**. In this collision kinetic energy is lost as a form of heat and sound during

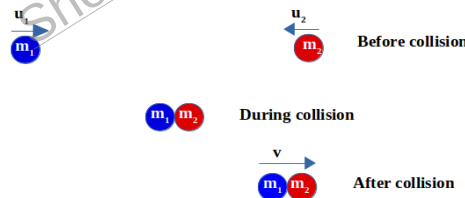


Figure 4.6: *perfectly inelastic collision*

collision. This lost of kinetic energy represented by Q-value. The Q-value is equal zero for elastic collision and less than zero ($Q < 0$) for inelastic collision. Coefficient of restitution (e) $= \frac{u_2 - u_1}{v_2 - v_1}$ is equal 1 for elastic collision and zero for inelastic collision.

Head-on Collisions: Collision, when objects rebound on straight line paths that co-incide with original direction of motion. These collisions can be treated one dimensionally.

Glancing Collisions: When Object do not collide on the same path line they make glancing collision. To solve this problem, break it into components as shown in figure

bellow.

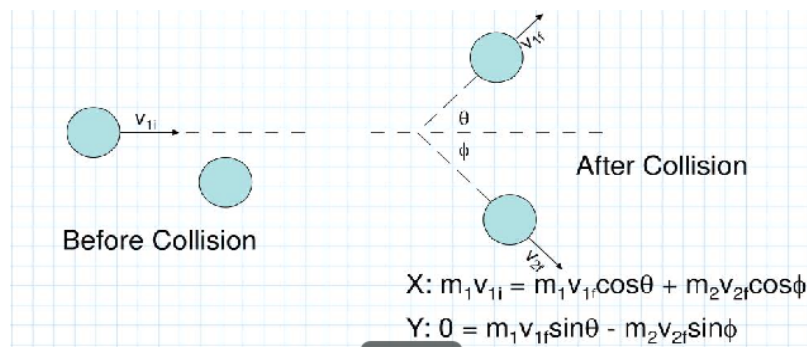


Figure 4.7: *Glancing Collision*

Class work

1. A ball of mass 2 kg is moving with a velocity of 12m/s collides with a stationary ball of mass 6 kg and comes to rest. Calculate velocity of ball of mass 6kg after collision.
2. A 10.0g bullet is fired into a stationary block of wood ($m = 5.00\text{kg}$). The bullet sticks into the block, and the speed of the bullet-plus combination immediately after collision is 0.600m/s. What was the original speed of the bullet?
3. A block of mass $m_1 = 1.6\text{kg}$ initially moving to the right with a speed of 4m/s on a horizontal frictionless track collides with a block of mass $m_2 = 2.1\text{kg}$ initially moving to the left with speed of 2.5m/s. If the collision is elastic, find the velocities of the two block after collision?
4. A partcle of mass 4.0kg initially moving with velocity of 2.0m/s collides with a partcle of mass 6.0kg, initially moving velocity of -4m/s. What are the velocity of the two particle after collision?
5. A 4kg block moving right at 6m/s collides elastically with a 2kg moving at 3m/s left, find final velocities the blocks.

4.3 Center of Mass and Moment of Inertia

4.3.1 Center of Mass

The center of mass of an object or system is the unique point at which the entire mass of the object or system can be considered to be concentrated. It is the point about which the object

or system will balance if it is supported at that point, and it is the point around which the object or system will rotate if it is free to do so. In a system of particles, the center of mass is the average position of all the particles in the system, weighted according to their masses. It is a useful concept in mechanics because it allows the analysis of the motion of an object or system as if all of its mass were concentrated at a single point. It is located somewhere on the line joining the particle and closer to the larger mass. The center of mass of an object or system can be found by taking the sum of the positions of all the particles in the system multiplied by their masses, and then dividing by the total mass of the system. Center of mass of several particles with mass $m_1, m_2, \dots, m_n$ at a distance $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ from each other is given by

$$\vec{r}_{cm} = \frac{\sum_{i=1}^n m_i \vec{r}_i}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n m_i \vec{r}_i}{M} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n}{m_1 + m_2 + \dots + m_n} \quad (4.12)$$

Where

$$M = \sum_{i=1}^n m_i = \text{total mass}$$

For coordinate x, y and z, center of mass given by

$$\vec{x}_{cm} = \frac{\sum_{i=1}^n m_i x_i}{M} \hat{i}, \quad \vec{y}_{cm} = \frac{\sum_{i=1}^n m_i y_i}{M} \hat{j} \quad \text{and} \quad \vec{z}_{cm} = \frac{\sum_{i=1}^n m_i z_i}{M} \hat{k} \quad (4.13)$$

This is known as the center of mass formula. The concept of center of mass is closely related to other important concepts in mechanics, such as the center of gravity, which is the point at which the gravitational force acting on an object or system can be considered to be concentrated. In many cases, the center of mass and the center of gravity of an object or system are at the same location, but this is not always the case, especially for objects or systems that are not symmetrical.

4.3.2 Moment of Inertia

Moment of Inertia: is the a measure of body's resistance to rotational motion about a particular axis. It is typically denoted by the symbol I and is measured in $kg * m^2$. Its magnitude is affected by distribution of mass of the body in relation to its axis of rotation. Thus there is no single value of moment of inertia of an object. But for a point mass moment of inertia is given by

$$I = mr^2$$

Moment of inertia of a rigid object, made up of a particles of mass $m_1, m_1, \dots$ at respective distance $r_1, r_2, \dots$ from its axis of rotation, its moment of inertial about that axis is given by

$$I = m_1 r_1^2 + m_2 r_2 + \dots m_i r_i^2 = \sum_i m_i r_i^2$$

For continuous mass distribution

$$I = \int_{m_0}^m r^2 dm$$

In general moment of inertia of a body depend on

- size of the body
- Shape of the body

For example consider disk and sphare of the same mass and the same radius.

$$I = \frac{1}{2} m r^2 \text{ for uniform disk}$$

$$I = \frac{2}{5} m r^2 \text{ for uniform sphare}$$

- It also depend on a point of axis of rotation

Example: For uniform rod axis of rotation through its center

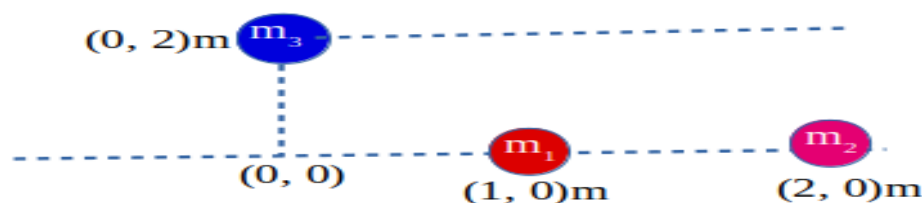
$$I = \frac{1}{12} M L^2$$

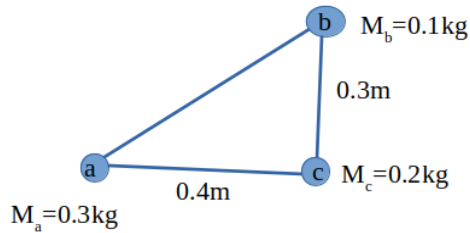
For uniform rod axis of rotation through its one end

$$I = \frac{2}{3} M L^2$$

Class work

1. A system consists three partcles of $m_1 = 1kg$, $m_2 = 1kg$ and $m_3 = 2kg$ located as in figure below. Find the center of mass of the system.





Find moment of inertia about an axis

- through point a perpendicular to the plane formed by the particle.
- along with rod bc

2. A particles are connected by a light rod as shown bellow
3. Calculate moment of inertia of 4 equal masses be each having a mass of 50g and situated at the corner of the square of side 30cm. when axis of rotation passes through
 - a) the center of the square perpendicular to the plane of the square
 - b) the center of the square perpendicular to sides of the square.
 - c) along the side of the square.

4.4 Torque and angular momentum

Torque: is defined as rotational effect of force ie (it is measure of force that cause an object to rotate around an axis. Torque is represented by symbol (greek letter) ' τ '. Torque is vector quality with both magintude and direction. It is calculated by the product of force and perpendicular distance from its axis of rotation.

$$\vec{\tau} = Fr \sin(\theta) \hat{n} = \vec{F} \times \vec{r}$$

Where θ is the direction of line of action of force $\hat{n}$ is unit vector in the direction of torque (clockwise or anticlockwise), r is moment arm of force (point of application of force from axis of rotation) F is applied.

Magintude of torque is depend on

- Size of force
- moment arm of force (radius or point of application of force)
- line of action of force (direction of line of action of force or θ)

Torque has maximum value when $\theta = 90^0$ and zero when $\theta = 0^0$ or 180^0

Torque intermes of angular accelarate From definition of Torue we have

$$\tau = Fr \perp$$

From Newton's 2nd law $F = a_T m$ for a particle moving in circle of radius but $a_T = \alpha r$ where α is angular acceleration Using those all together

$$\tau = \alpha r m r = \alpha (mr^2)$$

The term in the bracket is moment of inertia (I) so

$$\tau = I\alpha$$

Class work

1. A force of $\vec{F} = (2\hat{i} + 3\hat{j})$ is applied to an object that is pivoted about a fixed axis that is aligned along the z-axis. If the force is applied at a point located at $\vec{r} = (3\hat{i} - 2\hat{j})$

4.5 Conditions of Equilibrium (First and second)

Condition of Equilibrium in physics refers to state to the state where an object or system is not accelarating that is, its velocity is constant or zero (static equilibrium:-system is stable and at rest, the net torque must also be zero). There are two main conditions of equilibrium.

- 1 **First condition of equilibrium:** This states that, the net force acting object must be zero. This meanse that the vector sum of all force acting on the object must be equal to zero. Mathematically this can be expressed as

$$F_{net} = \sum F = \sum F_x + \sum F_y + \sum F_z = 0$$

- 2 **Second condition of equilibrium:** This states that, the net torque acting on the object must be zero. This meanse that the vector sum of all the torque acting on the object must be equal to zero. Mathematically this can be expressed as

$$\tau_{net} = \sum \tau = \sum \tau_{clockwise} + \sum \tau_{anticlockwise} = 0$$

Practical Example

Two kids balancing a seesaw satisfy both conditions for equilibrium. In the figure 4.8, we see

the lighter child sitting farther away from the pivot to create a torque equal in magnitude to that of the heavier child.

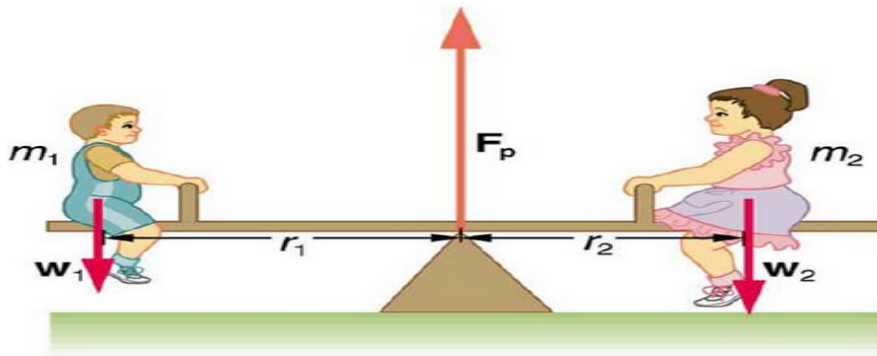


Figure 4.8: *seesaw satisfy both conditions for equilibrium*

Short reference material

Chapter 5

Work, Energy and Power

- Define work, kinetic energy and potential energy
- Calculate the work done by a constant force
- Derive work-kinetic energy theorem and apply in solving related problems
- State the principle of conservation of mechanical energy
- Solve problems related to the topics discussed in this section

5.1 Work done by constant and variable forces

Activity 5.1

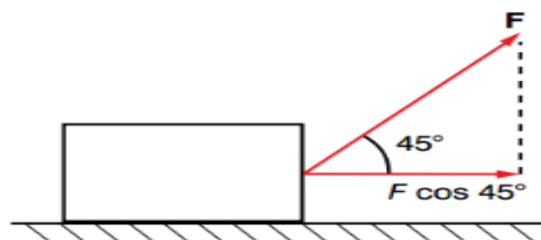
What is work done?

Work done is defined as the magnitude of the force exerted in the direction of the displacement (or distance moved) multiplied by the displacement. Therefore for work to be done on an object, three essential conditions should be satisfied:

- Force must be exerted on the object
- The force must cause a motion or displacement
- The force should have a component along the line of displacement

Both force and displacement moved are vectors. Work done is the scalar product of force and displacement:

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

Figure 5.1: *Forces on a free body diagram*

where F and d are the magnitudes of the vectors. Work done is a scalar - it does not have directional properties. We can show the forces on a free body diagram (Figure 5.1). The force in the direction of the displacement is $F \cos 45^\circ$, so work done is $W = Fd \cos \theta$ which is the equation for the scalar product of the force and displacement vectors. The work done by a force on a moving object is zero when the force applied is perpendicular to the displacement of its point of application. That is, if $\theta = 90^\circ$, then $W = 0$ because $\cos 90^\circ = 0$. The sign of the work also depends on the direction of F relative to d . The work done by the applied force is positive when the projection of F onto d is in the same direction as the displacement. When the projection of F onto d is in the direction opposite the displacement, W is negative. Remember that the unit of energy is the joule.

$$1 \text{ joule} = 1 \text{ Newton} \times 1 \text{ meter}$$

So 1 joule is the work done when a force of 1 Newton moves through a distance of 1 meter. This is the definition of the joule.

Example:

A man cleaning a floor pulls a vacuum cleaner with a force of magnitude $F = 50.0 \text{ N}$ at an angle of 30.0° with the horizontal (Fig. 5.2). Calculate the work done by the force on the vacuum cleaner as the vacuum cleaner is displaced 3.0m to the right

Figure 5.2: *A vacuum cleaner*

Solution

A free body diagram as shown in Figure 5.2b. Using the definition of work.

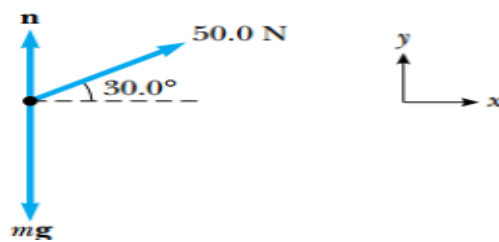


Figure 5.3: Free-body diagram

$$F = 50.0N, d = 3.0m, \theta = 30.0^\circ, \cos 30.0^\circ = 0.866$$

$$W = Fd\cos\theta = (50.0N)(3.00m)(\cos 30.0^\circ) = (50.0N)(3.00m)(0.866) = 130J$$

To finalize this problem, notice in this situation that the normal force n and the gravitational $F_g = mg$ do no work on the vacuum cleaner because these forces are perpendicular to its displacement.

Activity 5.2

What are the differences between work done by constant and variable forces?

5.1.1 Work done by a variable force

A force is said to perform work on a system if there is displacement in the system upon application of the force in the direction of the force. In the case of a variable force, integration is necessary to calculate the work done. The work done by a constant force of magnitude F , as we know, that displaces an object by Δx can be given as :

$$W = F \cdot \Delta x$$

In the case of a variable force, work is calculated with the help of integration. For example, in the case of a spring, the force acting upon any object attached to a horizontal spring can be given as:

$$F_s = -kx$$

Where, k is the spring constant, x is the displacement of the object attached. We can see that this force is proportional to the displacement of the object from the equilibrium position, hence the force acting at each instant during the compression and extension of the spring will

be different. Thus, the infinitesimally small contributions of work done during each instant are to be counted in order to calculate the total work done.

Therefore,

$$W_s = \int_{x_0}^x F_s(x) \cdot dx$$

$$W_s = - \int_{x_0}^x kx dx = -k \left[\frac{1}{2}x^2 \right]_{x_0}^x$$

$$W_s = -\left(\frac{1}{2}kx^2 - \frac{1}{2}kx_0^2\right) = -\frac{1}{2}k\Delta x^2$$

When a force varies, we cannot use the equation work done = force $\times$ distance moved. But the relationship for the area under the graph is still true. If we are able to record the force used and the displacement and plot a graph, we could find the work done by finding the area under the graph, as shown in Figure 5.3 You can estimate the average force by putting

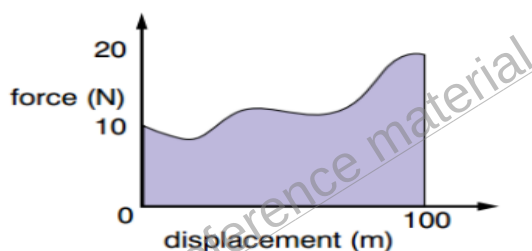


Figure 5.4: *Graph of variable force against displacement.*

a ruler on top of the graph as though you were going to draw a horizontal line. Adjust the position of the ruler so that the area between the graph line and the ruler is about the same above the ruler as it is below the ruler – this will give you an estimate of the average force

Example

A force $F = 2x + 5$ acts on a particle along the displacement. Find the work done by the force during the displacement of the particle from $x_0 = 0m$ to $x = 2m$. Given that the force is in Newton's.

Solution

Work done

$$W = \int_{x_0}^x F(x) \cdot dx = \int_{x_0}^x F(x) dx \cos(0) = \int_{x_0}^x F(x) dx$$

$$W = \int_0^2 (2x + 5) dx = \left[\frac{2x^2}{2} + 5x \right]_0^2 = (2 \times 2 + 5 \times 2) Nm = 14J$$

5.2 Conservation of energy

In physics the term work (or often work done) is another way of saying energy is being transferred from one object to another or transformed from one type to another.

$$\text{Work done} = \text{energy transferred}$$

The more energy transferred the more work has been done. We define the sum of kinetic and potential energies as mechanical energy:

$$E = U + KE$$

Where E is the total mechanical energy, KE is the kinetic energy and U is the all types of potential energy. So we can write the general form of the definition for mechanical energy without a subscript on U. In a system the mechanical energy of the system stays constant unless there is a force such as friction acting on the system.

The potential energy can be gravitational potential energy or energy stored in a spring, for example. When a spring is stretched, work is done because a force has been used to move one end of the spring by a certain displacement. Work is also done against gravity when you walk up stairs and you gain gravitational potential energy. When you walk down stairs, work is done by gravity and you lose gravitational potential energy.

We can show this as:

$$\text{Work done against gravity, } W = \Delta U = U_f - U_i$$

$$\text{Work done by gravity, } W = -\Delta U = -(U_f - U_i)$$

Where U_f Final potential energy and U_i initial potential energy

One of the possible outcomes of doing work on a system is that the system changes its speed. In this section, we investigate this situation and introduce our first type of energy that a system can possess, called kinetic energy.

$$W = \Delta K = K_f - K_i$$

The Conservation of mechanical energy is

$$E = \Delta U + \Delta KE$$

$$E = (U_f - U_i) + (K_f - K_i) = 0$$

$$(K_f + U_f = K_i + U_i \text{ (Conservation of mechanical energy)})$$

Example

A bullet weighing 20g is moving at a velocity of 500m/s. This bullet strikes a windowpane and passes through it. Now, its velocity is 400m/s. Calculate work done by a bullet when passing through this obstacle.

Solution

$$m = 20g = 0.02kg, v_1 = 500m/s, v_2 = 400m/s$$

We need to determine the change in kinetic energy in this equation. You know that kinetic energy change

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W = \frac{1}{2}m(v_2^2 - v_1^2)$$

$$W = \frac{1}{2} \times 0.02kg(400^2 - 500^2)m^2/s^2 = 0.01(160000 - 250000)kgm^2/s^2$$

$$W = -900J$$

This shows that the bullet lost energy of 900J ie work done by the bullet.

Activity 5.4

- (a) A boy walks up a hill. His displacement from his starting point is (800, 150) m. How much gravitational potential energy has he gained?

(b) The boy then walks to a village. The displacement from his starting point is (400, -50) m. How much gravitational potential energy did he lose going from the top of the hill to the village?

(c) What was the boy's net change in gravitational potential energy from his starting point to the village?
- A spring has a spring constant of 75 N/m. It is stretched by 20 cm. How much energy is stored in the spring?
- A force of 40 N is used to stretch a spring which has a spring constant of 350 N/m. How much energy is stored in the spring?
- A spring has a spring constant of 150 N/m and a mass is 100 g is attached to it. The spring sits on a horizontal frictionless surface and the other end of the spring is attached to a solid block. The mass is pulled by 10 cm to stretch the spring and then let go. What is the highest velocity of the mass?

5.3 Work energy theorem

According to Newton's second law of motion, the sum of all the forces acting on a particle,

$$F = ma$$

Let a force F is applied on an object initially moving with velocity u . If it is displaced to a displacement s and changes its velocity into v , then its motion will be expressed by the equation of motion. We can use the equation of motion

$$v^2 - v_0^2 = 2as$$

Multiplying this equation by mass m and dividing throughout by 2, we get:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mas = Fs = W$$

$$\therefore W = \Delta KE$$

Example

A car with a mass of 1,000 kg brakes to a stop from a velocity of 20 m/s (45 mi/hr) over a length of 50 meters. What is the force applied to the car? Solution

$$\Delta KE = 0 - [(1/2)(1,000 \text{ kg})(20 \text{ m/s})^2] = -200,000 \text{ J}$$

$$W = -200,000 \text{ Nm} = (F)(50 \text{ m}) \Rightarrow F = -4,000 \text{ N}$$

Activity 5.5

1. A football of mass 550 g is at rest on the ground. The football is kicked with a force of 108 N. The footballer's boot is in contact with the ball for 0.3 m.
 - a) What is the kinetic energy of the ball?
 - b) What is the ball's velocity at the moment it loses contact with the footballer's boot?
2. A car of mass 1200 kg accelerates from 5 m/s to 15 m/s. The force of the engine acting on the car is 6000 N. Over what distance did the force act?

5.4 Conservative forces

Conservative forces a force that does not work when a body moves on a closed path. Conservative forces have these two equivalent properties:

- 1 The work done by a conservative force on a particle moving between any two points is independent of the path taken by the particle.
- 2 The work done by a conservative force on a particle moving through any closed path is zero. (A closed path is one in which the beginning and end points are identical.) The gravitational force is one example of a conservative force, and the force that a spring exerts on any object attached to the spring is another.

Activity 5.5

1. What are the differences between conservative and dissipative forces?

5.5 Power

Power is the rate at which work is done, or the work done per second. It is measured in the units joules per second (j/s), which are also called watts (W).

$$Power = \frac{\text{total work done}}{\text{total time taken}}$$

Example:

A garage hoist lifts a truck up 2 meters above the ground in 15 seconds. Find the power delivered to the truck. [Given: 1000 kg as the mass of the truck] **Solution**

First we need to calculate the work done, which requires the force necessary to lift the truck against gravity:

$$F = mg = 1000 \times 9.81 = 9810 \text{ N.}$$

$$W = Fd = 9810\text{N} \times 2\text{m} = 19620 \text{ Nm} = 19620 \text{ J.}$$

$$\text{The power is } P = W/t = 19620\text{J} / 15\text{s} = 1308 \text{ J/s} = 1308 \text{ W.}$$

Activity 5.6

1. A weightlifter lifts 200 kg through 1.8 m in 2 s.
 - a) What is the weightlifter's power?
 - b) Why is his actual power likely to be higher than this?

2. A petrol engine raises 200 liters of water in a well from a depth of 7 m in 6 seconds. Show that the power of the engine is about 2330 W.
3. Look at question 1 on page 101. It takes 4 seconds to drag the container up the slope. What is the power?
4. Look at question 2 on page 101. The man takes 12 seconds to drag the box. What is his power?
5. A spring with a spring constant of 275 N/m is stretched 20 cm in 2 seconds. What is the power applied to stretch the spring?

Review questions

1. A football of mass 550 g is at rest on the ground. The football is kicked with a force of 108 N. The footballer's boot is in contact with the ball for 0.3 m.
 - a. What is the kinetic energy of the ball?
 - b. What is the ball's velocity at the moment it loses contact with the footballer's boot?
2. A car of mass 1200 kg accelerates from 5 m/s to 15 m/s. The force of the engine acting on the car is 6000 N. Over what distance did the force act?
3.
 - a) A boy walks up a hill. His displacement from his starting point is (800, 150) m. How much gravitational potential energy has he gained?
 - b) The boy then walks to a village. The displacement from his starting point is (400, -50) m. How much gravitational potential energy did he lose going from the top of the hill to the village?
 - c) What was the boy's net change in gravitational potential energy from his starting point to the village?
4. A spring has a spring constant of 75 N/m. It is stretched by 20 cm. How much energy is stored in the spring?
5. A force of 40 N is used to stretch a spring which has a spring constant of 350 N/m. How much energy is stored in the spring?
6. A spring has a spring constant of 150 N/m and a mass of 100 g is attached to it. The spring sits on a horizontal frictionless surface and the other end of the spring is attached to a solid block. The mass is pulled by 10 cm to stretch the spring and then let go. What is the highest velocity of the mass?

7. A ball of mass 500 g is kicked into the air at an angle of 45° . It reaches a height of 12 m. What was its initial velocity?
8. A pendulum bob has a mass of 1 kg. The length of the pendulum is 2 m. The bob is pulled to one side to an angle of 10° from the vertical.
 - a) What is the velocity of the pendulum bob as it swings through its lowest point?
 - b) What is the angular velocity of the pendulum bob?
9. A weightlifter lifts 200 kg through 1.8 m in 2 s.
 - a) What is the weightlifter's power?
 - b) Why is his actual power likely to be higher than this?
10. A petrol engine raises 200 liters of water in a well from a depth of 7 m in 6 seconds. Show that the power of the engine is about 2330 W.
11. . Look at question 1 on page 101. It takes 4 seconds to drag the container up the slope. What is the power?
12. Look at question 2 on page 101. The man takes 12 seconds to drag the box. What is his power?
13. . A spring with a spring constant of 275 N/m is stretched 20 cm in 2 seconds. What is the power applied to stretch the spring?
14. . How can you derive the work–energy theorem from Newton's second law of motion?

Chapter 6

Oscillation and Waves

Objective: - At the end of this unit students should be able to:

- Describe the periodic motion of a vibrating object in qualitative terms, and analyse it in quantitative terms (e.g. the motion of a pendulum, a vibrating spring, a tuning fork).
- Define simple harmonic motion (SHM) and describe the relationship between SHM and circular motion.
- Derive and use expressions for the frequency, periodic time, displacement, velocity and acceleration of objects performing SHM.
- Describe the effects: free oscillations, damping, forced oscillations and resonance.
- Explain the energy changes that occur when a body performs SHM.
- Relate the energy of an oscillator to its amplitude.
- Solve problems on SHM involving period of vibration and energy transfer.
- Describe the characteristics of a mechanical wave and identify that the speed of the wave depends on the nature of the medium.
- Calculate the frequency of the harmonics along a string, an open pipe and a pipe closed at one end.

6.1 Oscillatory motion

Simple harmonic motion the periodic oscillation of an object about an equilibrium position, such that its acceleration is always directly proportional in size but opposite in direction

to its displacement oscillating (vibrating) about a central position equilibrium position the position of an oscillating object when at rest restoring force the force on a displaced object that acts towards its original position.

When a body repeats its path of motion back and forth about the equilibrium or mean position, the motion is said to be periodic. All periodic motions need not be back and forth like the motion of the earth about the sun, which is periodic but not vibratory in nature. The periodic motion in which there is existence of a restoring force and the body moves along the same path to and fro about a definite point, equilibrium position, is called oscillatory motion

6.1.1 Harmonic Motion

Any motion that repeats at regular intervals is called periodic motion or harmonic motion. However, here we are interested in a particular type of periodic motion called simple harmonic motion (SHM).

Periodic oscillations

If something is oscillating (vibrating) this means that it is moving backwards and forwards, up and down, side to side, in and out, etc, around some central position. This central position is called the equilibrium position and it is the position of the object when it is at rest.

Whenever an object is displaced from its equilibrium position there is a force that acts towards its original position. This force is often referred to as a restoring force, as it tries to restore the system to its equilibrium position. This is much easier to understand if we look at some simple examples.

How does a pendulum work?

A simple pendulum is made by hanging a mass, known as the bob, on a string from a fixed support, as shown in Figure 6.1. If we let the mass hang without swinging, it will hang directly below the support with all forces on it balanced. This position, where the resultant force acting on the bob is zero, is known as the equilibrium position.

If we give the bob a small initial displacement by pulling it to one side and then release it, there will be a resultant force, due to the weight of the bob and the tension acting in the string. This force pulls it back towards the equilibrium position. This causes acceleration towards the equilibrium position (opposite to the direction of displacement).

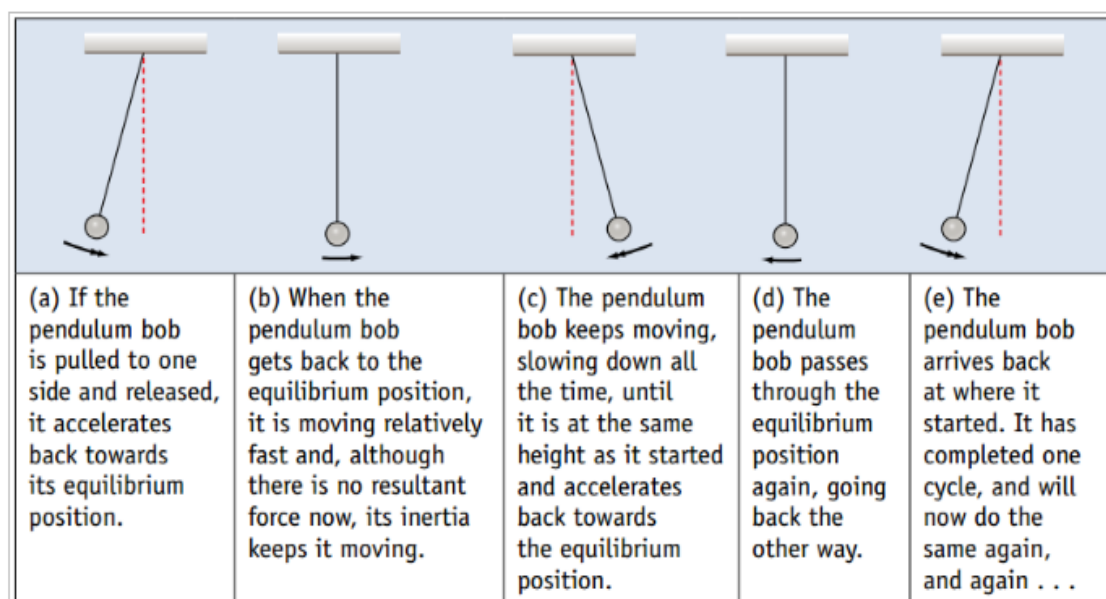


Figure 6.1: *Oscillation of a pendulum when the bob is pulled to one side and released*

When the bob reaches the equilibrium position, the resultant force is now zero, but the bob is moving and can't stop instantly. Its inertia keeps it moving through the equilibrium position, and if there is no significant friction or air resistance, it will keep moving, slowing down all the time until it is as high as it was when it started.

It now has a displacement equal and opposite to its starting displacement. However, as displacement is a vector quantity it is now a negative value. If the initial displacement was 3 cm, the displacement after one swing (half an oscillation) will be -3 cm.

In exactly the same way, it will swing back to where it started to complete one complete cycle of the oscillation. It will now repeat this process again and again. It is important to notice the force causing the oscillation always acts towards the equilibrium position.

How does a mass on a spring oscillate?

If a mass is hung from a support by a spring and allowed to settle until it is stationary, it will hang with the spring stretched so that the restoring force (in this case the tension in the spring) is equal and opposite to the weight of the mass. This is the equilibrium position.

If we now pull the mass down, the tension in the spring will be greater than the weight of the mass. The resultant force on the mass is upwards and so, if we let go, it accelerates

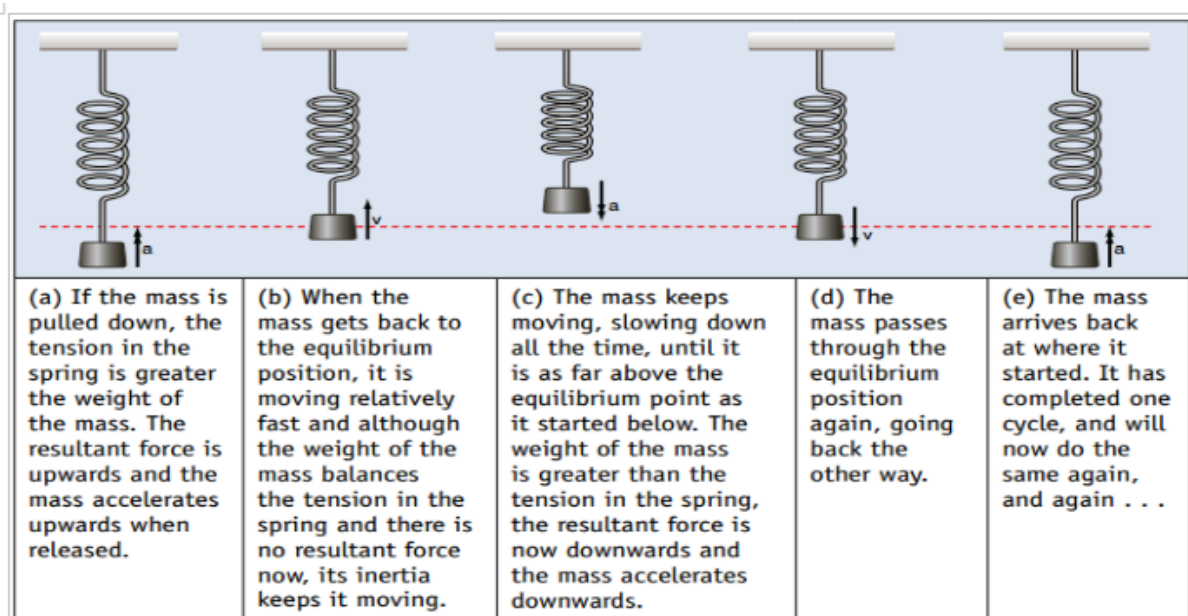


Figure 6.2: *Oscillation of a mass-spring system when the mass is displaced downwards and released*

upwards. When the mass gets back to the equilibrium position it is moving and, although there is no resultant force here, its inertia keeps it moving.

The mass keeps moving, slowing down all the time, until it is as far above the equilibrium point as it started below. The tension in the spring is now less than the weight of the mass, the resultant force is now downwards and the mass accelerates downwards. The mass passes through the equilibrium position again, and carries on until it arrives back at where it started. It has completed one cycle, and will now do the same again, and again.

What does SHM look like?

If we plot how the displacement of an object performing simple harmonic motion varies with time, we find that the variation is sinusoidal, as shown in Figure 6.3. Note that the displacement goes positive and negative as the mass oscillates either side of the equilibrium position.

The size of the maximum displacement in either direction is called the amplitude A . The time to perform one complete cycle of the oscillation is called the time period T . When we say the oscillation is sinusoidal, we mean that the displacement is described mathematically

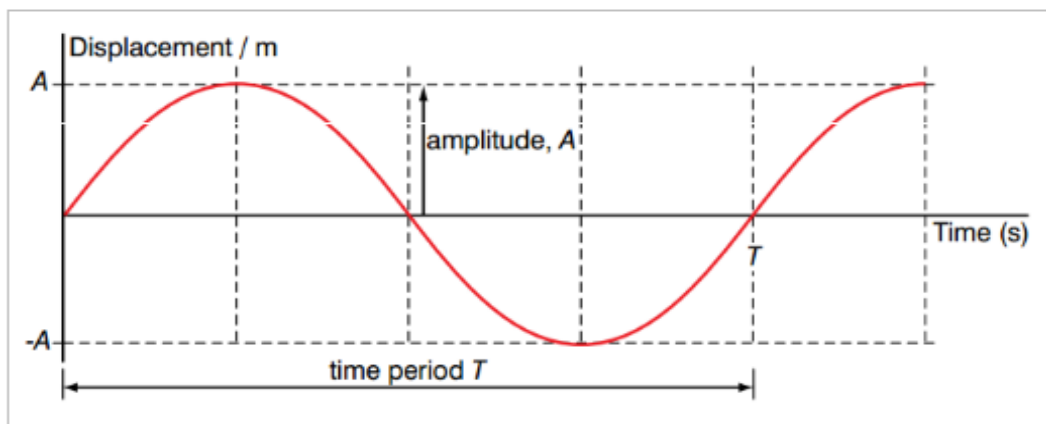


Figure 6.3: Variation of displacement with time for simple harmonic motion

using sine or cosine functions:

$$x = A \sin\left(2\pi \frac{t}{T}\right) \text{ or } x = A \cos\left(2\pi \frac{t}{T}\right) \quad (6.1)$$

where A is the amplitude of the oscillation and T the time period. Either could be used, but throughout the rest of this chapter we will use,

$$x = A \sin\left(2\pi \frac{t}{T}\right) \quad (6.2)$$

although the cosine function gives a better description if the SHM is started by displacing the oscillator and then releasing it.

If $x = A \sin(2\pi \cdot \frac{t}{T})$, with $(\frac{2\pi}{T} \cdot t)$ in the expression relation.

When $t = 0$,

$$x = A \sin\left(\frac{2\pi}{T} \cdot 0\right) = A \sin(0) = 0$$

for $t = \frac{T}{4}$

$$x = A \sin\left(\frac{2\pi}{T} \cdot \frac{T}{4}\right) = A \sin\left(\frac{\pi}{2}\right) = A$$

for $t = \frac{T}{2}$

$$x = A \sin\left(\frac{2\pi}{T} \cdot \frac{T}{2}\right) = A \sin(\pi) = 0$$

for $t = \frac{3T}{4}$

$$x = A \sin\left(\frac{2\pi}{T} \cdot \frac{3T}{4}\right) = A \sin\left(\frac{3\pi}{2}\right) = -A$$

for $t = T$

$$x = A \sin\left(\frac{2\pi}{T} \cdot T\right) = A \sin(2\pi) = 0$$

Looking carefully at the information above you can see how in one oscillation the displacement starts at 0 rises to a positive amplitude, falls back to zero, falls to a negative amplitude and then rises back to zero.

6.1.2 Damped and Forced Oscillation

Damped Oscillation

The oscillatory motions we have considered so far have been for ideal systems—that is, systems that oscillate indefinitely under the action of only one force a linear restoring force. In many real systems, non conservative forces, such as friction, retard the motion.

Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be damped. Figure 6.4 depicts one such system: an object attached to a spring and submersed in a viscous liquid. One common type of retarding force is proportional to the

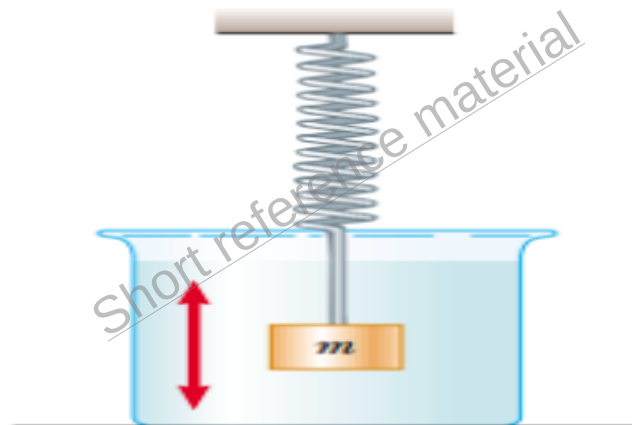


Figure 6.4: *One example of a damped oscillator is an object attached to a spring and submersed in a viscous liquid*

speed of the moving object and acts in the direction opposite the motion. This retarding force is often observed when an object moves through air, for instance.

Air resistance and friction are typical examples of damping forces and are the reason why pendulums naturally stop swinging and masses on springs stop oscillating. The damping force is given by:

$$F_D = -bv \quad (6.3)$$

b = the damping coefficient and is dependent on the medium providing the damping, v = the velocity of the object through the medium.

This equations shows how the resistive force is directly proportional, but opposite, to the velocity. As a result the amplitude of the oscillation will decay exponentially, as shown overleaf in Figure 6.5 (a). Note that the period of the oscillation does not change as the amplitude gets smaller. Heavier damping causes a more rapid decay of amplitude as shown in Figure 6.5(b). Damping in a car suspension is not normally so heavy, as this would produce

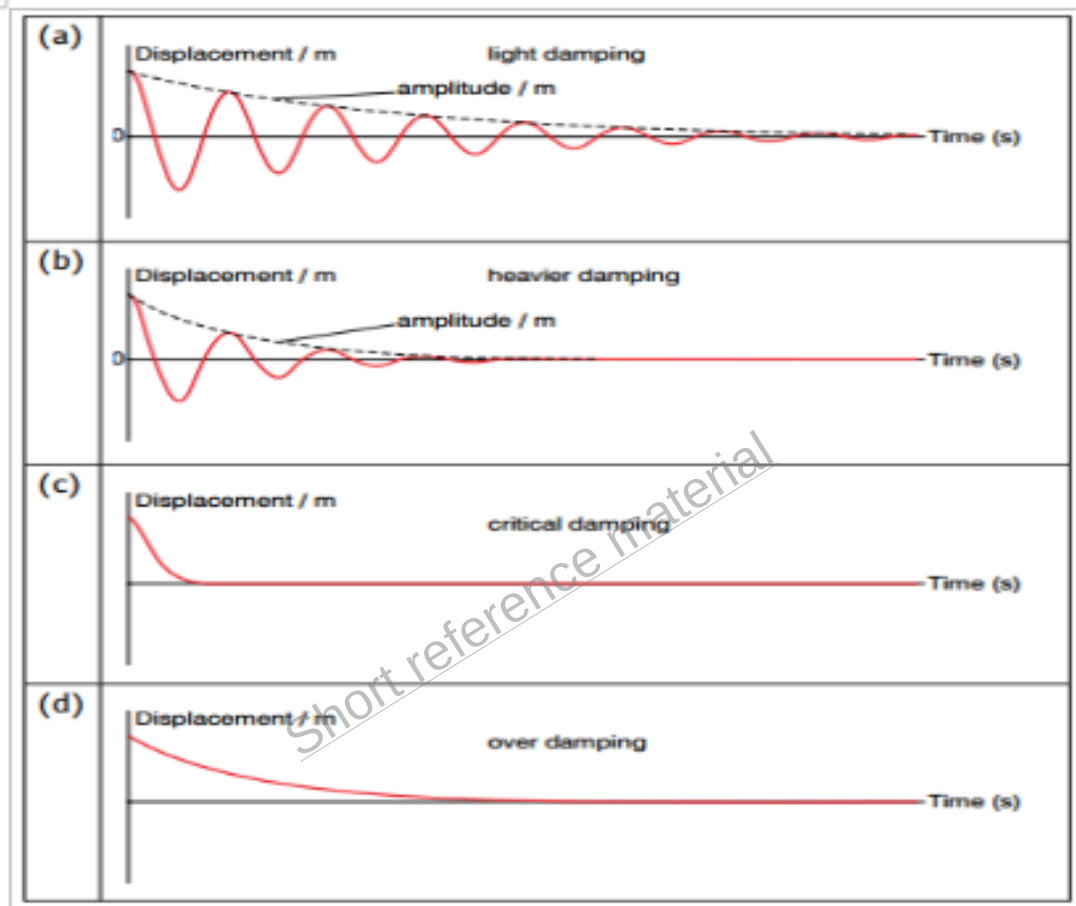


Figure 6.5: *Plots of displacement against time for an oscillator that is displaced and then released, for different amounts of damping.*

a very ‘hard’ and uncomfortable ride for the passengers. The damping shown in Figure 6.5(b), on the other hand, would provide a very bouncy ride; this would be called **underdamping**.

The damping in a car suspension is always a compromise somewhere near to the critical damping shown in Figure 6.5(c). Critical damping is the amount of damping that leads to the oscillator settling back to a stationary state at the equilibrium position in the shortest possible time.

Damping reduces the effects of resonance. As the periodic driving force transfers energy into the oscillator the damping mechanism dissipates the energy. The resonance peak in the graph of driven amplitude against driving frequency becomes lower and relatively wider, as shown in Figure 6.6. It can also be seen that damping also causes a very small reduction in the natural frequency of the oscillator.

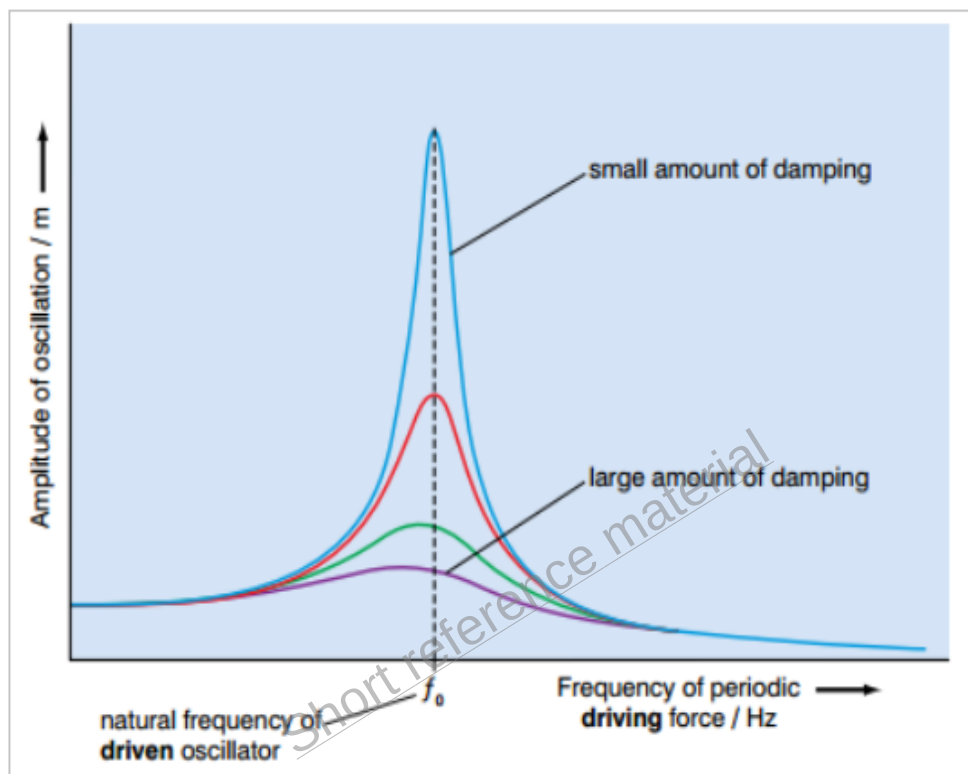


Figure 6.6: *Driven amplitude against driving frequency for forced oscillations of an oscillator with different amounts of damping*

Forced Oscillation

A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as $F(t) = F_0 \sin \omega t$, where ω is the angular frequency of the driving force and F_0 is a constant. In general, the frequency of the driving force is variable while the natural frequency ω of the oscillator is fixed by the values of k and m . Newton's second law in this situation gives

$$\sum F = ma \leftrightarrow F_0 \sin \omega t - b \frac{dx}{dt} - kx = m \frac{d^2x}{dt^2} \quad (6.4)$$

After a sufficiently long period of time, when the energy input per cycle from the driving force equals the amount of mechanical energy transformed to internal energy for each cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude.

$$A \sin(\omega t + \phi)$$

Where

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

and where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the undamped oscillator ($b = 0$).

For small damping, the amplitude is large when the frequency of the driving force is near the natural frequency of oscillation, or when 0. The dramatic increase in amplitude near the natural frequency is called resonance, and the natural frequency 0 is also called the resonance frequency of the system.

6.2 Properties of wave (frequency, wave length, period)

Terminologies in Wave

Crests/Troughs: are positions in a wave with maximum displacements above/below the equilibrium position.

Amplitude (A): is the maximum displacement from the equilibrium position.

Displacement (y): is position of a wave from equilibrium position at any time.

Wave length (λ): distance between any two consecutive points which are in phase.

Period (T): is the time taken by a wave to move one wave length.

Frequency (f): number of oscillations performed per unit time.

Speed (v): is constant in a medium provided the medium is homogeneous.

What is a travelling wave?

Electromagnetic and sound waves are particularly important to us, but waves on water are a little easier to observe. A travelling wave transfers energy, and sometimes information, from

one place to another, in what is called the direction of propagation. An oscillation at the source of energy causes an oscillation to travel through space. For electromagnetic waves this oscillation is of electric and magnetic fields and does not need a medium. In a mechanical wave that involves the oscillations of particles of a physical medium, as the particles pass on energy, they undergo temporary displacements but no permanent change in the position. For example, when ripples travel across a pond the water molecules oscillate vertically but do not move in the direction of the wave.

Frequency and time period

The frequency, f , of an oscillation is the number of cycles it completes per second. The unit is the hertz, symbol Hz. A frequency of 50 Hz would correspond to 50 complete oscillations per second. Frequency is related to time period by:

$$f = \frac{1}{T}$$

and so our mathematical expression for displacement can be written as

$$x = A \sin(2\pi ft)$$

Activities

1. An object moving with simple harmonic motion has an amplitude of 3 cm and a frequency of 30 Hz. Calculate:
 2. the time period of the oscillation,
 3. the acceleration in the centre and at the maximum displacement of an oscillation, and
 4. the velocity in the centre and at the maximum displacement of an oscillation
5. Describe the key features of the different forms of damping the general effect of damping on resonance.

6.3 Types of Waves

6.3.1 Transverse and longitudinal

Waves can also be categorized as transverse and longitudinal waves based on the way they are propagating.

1. **Transverse Wave-** is a wave where particles of the disturbed medium oscillate perpendicular to the direction of wave motion. Examples are: water waves, waves on strings, and all EM waves. Sinusoidal graphs can represent this motion.
2. **Longitudinal Wave-** is a wave where particles of the disturbed medium oscillate parallel to the direction of wave motion. Example: sound wave

6.3.2 Mechanical and Electromagnetic wave

Waves can be categorized as Mechanical and Electromagnetic waves based on the need of material medium for its propagation.

1. **Mechanical Waves-** are waves produced by the oscillation of particles of a mechanical medium and need a medium for propagation. Examples are water waves, sound wave, waves in strings etc.

All mechanical waves require:

- some source of disturbance
- a medium that can be disturbed and
- physical medium through which elements of the medium can influence each other.

2. **Electromagnetic (EM) waves:-**are produced by accelerated charged particles and can propagate through both material medium and vacuum. Examples are: Light, radio and television waves, micro waves, x-rays, etc. All EM waves in vacuum propagate with speed $c = 3.0 \times 10^8 \text{ m/s}$.

Waves can either move in space (e.g water waves), the so called traveling waves, or be stationary in an enclosure, the so called standing waves.

6.4 Wave behavior (reflection, refraction, interference, diffraction)

The characteristics of waves are important in determining the size of waves, the speed at which they travel, how they break on shore, and much more. Following are some of the characteristics of waves.

Reflection of Waves

Whenever a traveling wave reaches a boundary, part or all of the wave bounces back. This phenomenon (rebounding of wave from a surface) is called reflection. For example, consider a pulse traveling on a string that is fixed at one end. When the pulse reaches the wall, it is reflected.

Refraction of wave

It is the change in direction of a wave passing from one medium to another caused by its change in speed. For example, waves in deep water travel faster than in shallow. If an ocean wave approaches a beach obliquely, the part of the wave farther from the beach will move faster than that closer in, and so the wave will swing around until it moves in a direction perpendicular to the shoreline. The speed of sound waves is greater in warm air than in cold. At night, air is cooled at the surface of a lake, and any sound that travels upward is refracted down by the higher layers of air that still remain warm. Thus, sounds, such as voices and music, can be heard much farther across water at night than in the daytime.

Diffraction of wave

It is the spreading of waves around obstacles. Diffraction takes place with sound; with electromagnetic radiation, such as light, X-rays, and gamma rays; and with very small moving particles such as atoms, neutrons, and electrons, which show wavelike properties. One consequence of diffraction is that sharp shadows are not produced. The phenomenon is the result of interference (i.e., when waves are superimposed, they may reinforce or cancel each other out) and is most pronounced when the wavelength of the radiation is comparable to the linear dimensions of the obstacle.

Interference of wave

It is the net effect of the combination of two or more wave trains moving on intersecting or coincident paths. The effect is that of the addition of the amplitudes of the individual waves at each point affected by more than one wave.

Interference also occurs between two wave trains moving in the same direction but having different wavelengths or frequencies. The resultant effect is a complex wave. A pulsating frequency, called a beat, results when the wavelengths are slightly different.

6.5 Wave equation

The frequency of a wave can be defined in two equivalent ways. It is the frequency of the individual oscillators that pass the energy along, the number of times particles go up and down or backwards and forwards per second. It is also the number of complete waves, the number of wavelengths that pass any given point per second. If the wavelength is λ , and f wavelength pass a point per second, then the speed of the wave must be given by the wave equation:

$$v = \lambda f \quad (6.5)$$

The speed of any travelling wave depends on the media it is travelling. For a mechanical wave travelling along a string the speed of the wave depends on the tension of the string and the mass per unit length (sometimes called linear density).

$$v = \sqrt{\frac{T}{\mu}}$$

where μ = mass per unit length given by $\mu = m/l$ in kg/m T = tension in the string in N.

The formula given above shows us that the 'tighter' the string the faster the waves will travel down its length. Additionally the 'lighter' the string, (the smaller its mass/length ratio), the faster the waves will travel down its length. The **phase speed** of a wave is the rate at which the phase of the wave travels through space. Any given phase of the wave (for example, the crest or the trough) will appear to travel at the phase velocity. The phase velocity is given in terms of the wavelength λ (lambda) and period T as

$$v_{\text{phase}} = \frac{\lambda}{T} \quad (6.6)$$

Review questions

1. A simple pendulum is made from a bob of mass 0.040 kg suspended on a light string of length 1.4 m. Keeping the string taut, the pendulum is pulled to one side until it has gained a height of 0.10 m. Calculate
 - a) the total energy of the oscillation
 - b) the amplitude of the resulting oscillations
 - c) the period of the resulting oscillations
 - d) the maximum velocity of the bob

- e) the maximum kinetic energy of the bob.
2. A piston in a car engine has a mass of 0.75 kg and moves with motion which is approximately simple harmonic. If the amplitude of this oscillation is 10 cm and the maximum safe operating speed of the engine is 6000 revolutions per minute, calculate:
- maximum acceleration of the piston
 - maximum speed of the piston
 - the maximum force acting on the piston constant?
3. A car of mass 820 kg has an under damped suspension system. When it is driven by a driver of mass 80 kg over a long series of speed bumps 10 m apart at a speed of 3 m/s the car bounces up and down with surprisingly large amplitude.
- Explain why this effect occurs.
 - Calculate the net spring constant of the car suspension system.
4. If you are given a metal rod and a hammer, how must you hit the rod to produce:
- a transverse wave, and
 - a longitudinal wave?
5. A whistle producing a sound at 1 KHz is whirled in a horizontal circle at a speed of 18 m/s. What are the highest and lowest frequencies heard by a listener standing a few metres away, if the speed of sound in air is 340 m/s? 6. If the speed of sound in air is 340 m/s, what is the wavelength of a sound wave at 512 Hz?
6. A travelling wave on a string, of amplitude 2 mm, frequency 500 Hz and speed 300 m/s, can be described by the function
7. $Y = A \sin(2\pi - 2\pi ft)$
- Sketch graphs of displacement Y against distance x for this wave, for the first 1.2 m:
 - for time $t = 0$, and
 - for time $t = 0.5$ ms
 - Sketch graphs of displacement Y against time t for the oscillation produced by this wave for the first 4 ms

- i) at the source where $x = 0$, and
- ii) at a distance $x = 30$ cm from the source.

Short reference material

Short reference material

Chapter 7

Heat and thermodynamics

Objective: - By the end of this unit students should be able to:

- Define the zeroth law of thermodynamics.
- Determine the relationship between temperature and energy transfer and thermal equilibrium.
- Give the definitions of isothermal, isobaric, isochoric and adiabatic processes.
- State the first law of thermodynamics.
- Describe ways of changing the internal energy of a gas.
- Solve problems involving calculations of pressure, temperature or volume for a gas undergoing adiabatic changes.
- State the assumptions made to define an ideal gas.
- Describe the kinetic theory of gases, including the importance of Brownian motion and diffusion.

7.1 Temperature and Heat

Thermodynamics is a science of the relationship between heat, work, temperature, and energy. Temperature is more difficult to define and we will encounter a number of different ways to approach of temperature.

Temperature is something that we all have experience of. If we place two bodies of different temperatures in contact, then the particles at the boundary will collide and the kinetic energy of particles is transferred backwards and forwards between the objects. A ‘body’ is another word for an object. On average, the particles in the hotter body have more kinetic energy than those in the colder body, so there is a net transfer of thermal energy from the hotter body to the colder body.

This process is referred to as heating. This is the only way that the word heat can be used. A body does not contain or possess heat. This is just the same as an electrical component, which does not contain or possess electrical current. Instead we will use the term internal energy to describe the total energy that is internal to bodies.

Temperature is a measure of the average random kinetic energy of particles in a body, and is used to determine in which direction there will be a net energy flow when two bodies are close to one another.

Temperature Scales

Thermometers measure temperature according to well-defined scales of measurement. The three most common temperature scales are Fahrenheit, Celsius, and Kelvin. Temperature scales are created by identifying two reproducible temperatures. The freezing and boiling temperatures of water at standard atmospheric pressure are commonly used. On the Celsius

	Absolute Temperature	Celsius Temperature
Absolute zero	0.00	-273.15
Triple point of water	273.16	0.01
Ice point	273.15	0.00
Steam point	373.15	100.00
Room Temperature	293	20

scale, the freezing point of water is 0°C and the boiling point is 100°C . The unit of temperature on this scale is the degree Celsius ($^{\circ}\text{C}$). The Fahrenheit scale has the freezing point of water at 32°F and the boiling point at 212°F . Its unit is the degree Fahrenheit ($^{\circ}\text{F}$).

Example

”Room temperature” is generally defined in physics to be 25°C . (a) What is room temperature in $^{\circ}\text{F}$? (b) What is it in K?

To convert from	Use this Equation
Celsius to Fahrenheit	$T_F = \frac{9}{5}T_c + 32$
Fahrenheit to Celsius	$T_c = \frac{5}{9}(T_F - 32)$
Celsius to kelvin	$T_k = T_c + 273.15$
kelvin to Celsius	$T_c = T_k - 273.15$
Kelvin to Fahrenheit	$T_F = \frac{9}{5}(T_k - 273.15) + 32$

Solution

To convert from $^{\circ}\text{C}$ to $^{\circ}\text{F}$, use the equation

$$T_F = 9/5T_C + 32$$

Substitute the known value into the equation and solve:

$$T_F = 9/5(25^{\circ}\text{C}) + 32 = 77^{\circ}\text{F}$$

. Similarly, we find that

$$T_K = T_C + 273.15 = 298\text{K} = 298\text{K}$$

.

Activity

1 Convert the following to degrees Celsius:

a) the boiling point of helium, 4.25 K

b) the freezing point of gold, 1340 K.

2 Convert the following to kelvin:

a) the freezing point of mercury, -39°C

b) the average temperature of the universe, -270.42°C .

7.2 The effect of heat on materials (change of Temperature, expansion, change of phase, heat capacity)

Thermal Expansion

The expansion of alcohol in a thermometer is one of many commonly encountered examples of thermal expansion, which is the change in size or volume of a given system as its temperature changes. The most visible example is the expansion of hot air. When air is heated, it expands and becomes less dense than the surrounding air, which then exerts an (upward) force on the hot air and makes steam and smoke rise, hot air balloons float, and so forth.

Linear Thermal Expansion

The increase in length ΔL of a solid is proportional to its initial length L_0 and the change in its temperature ΔT . The proportionality constant is called the coefficient of linear expansion, α .

$$\Delta L = \alpha L_0 \Delta T \quad (7.1)$$

$$L = L_0 + \Delta L = L_0(1 + \alpha \Delta T) \quad (7.2)$$

$$\alpha = \frac{\Delta L}{L_0 \Delta T}$$

And has unit of K^{-1} or $^{\circ}C^{-1}$

Table 7.1: **Some typical coefficients of thermal expansion.**

Substance	Coefficient of Linear expansion, $\alpha(K^{-1})$
Lead	29×10^{-6}
Aluminium	24×10^{-6}
Brass	19×10^{-6}
Copper	17×10^{-6}
Iron (steel)	12×10^{-6}
Concret	12×10^{-6}
Window glass	11×10^{-6}
Pyrex glass	3.3×10^{-6}
Quartz	0.50×10^{-6}

7.2. THE EFFECT OF HEAT ON MATERIALS (CHANGE OF TEMPERATURE, EXPANSION, C

Areal Expansion

The change in area ΔA of a solid is proportional to its initial area A_0 and the change in its temperature ΔT . That is,

$$\Delta A = \beta A_0 \Delta T \quad (7.3)$$

$$A = A_0 + \Delta A = A_0(1 + \beta \Delta T) \quad (7.4)$$

Where $\beta = 2\alpha$ is coefficient of areal expansion

Volume Expansion

The change in volume ΔV of a solid is proportional to its initial volume V_0 and the change in its temperature ΔT . That is:

$$\Delta V = \gamma V_0 \Delta T \quad (7.5)$$

$$V = V_0 + \Delta V = V_0(1 + \gamma \Delta T) \quad (7.6)$$

Where $\gamma = 3\alpha$ is coefficient of volume expansion

Table 7.2: Some typical coefficients of thermal expansion.

Substance	Coefficient of Volume expansion, $\gamma(K^{-1})$
Ether	1.51×10^{-3}
Carbon Tetrachloride	1.18×10^{-3}
Alcohol	1.01×10^{-3}
Gasolin	17×10^{-3}
Olive Oil	0.95×10^{-3}
Water	0.21×10^{-3}
Mercury	0.18×10^{-3}

Example

1. A steel rod has a length of exactly 20 cm at $30^{\circ}C$. How much longer is it at $50^{\circ}C$?

[Use $\alpha_{Steel} = 11 \times 10^{-6}/C$.]

Solution The change in temperature of the steel rod is

$$\Delta T = 50^{\circ}C - 20^{\circ}C = 30^{\circ}C$$

and the length is 20.0 cm. Using the given value for the linear expansion coefficient α ,

we find the change in length from Equation of expansion,

$$\Delta L = L_0 \alpha \Delta T = (20.0 \text{ cm})(11 \times 10^{-6} / ^\circ \text{C})(20.0^\circ) = 4.4 \times 10^{-3} \text{ cm}$$

The length of the bar increases by $4.4 \times 10^{-3} \text{ cm}$.

2. By how much does the volume of an aluminum cube 5.00 cm on an edge increase when the cube is heated from 10.0°C to 60.0°C ? [Use $\alpha_{\text{Al}} = 23 \times 10^{-6} / ^\circ \text{C}$]

Solution

From the given value of α (the linear expansion coefficient) and Equation of areal expansion, we can get β , the volume expansion coefficient:

$$\beta = 3\alpha = 3(23 \times 10^{-6} / ^\circ \text{C}) = 69 \times 10^{-6} / ^\circ \text{C}$$

Now, the volume of the cube is $V = (5.00 \text{ cm})^3 = 125 \text{ cm}^3$; this is really the initial volume, and as usual we don't expect it to change much. The change in temperature of the cube is $\Delta T = 50.0^\circ \text{C}$. the corresponding increase in volume:

$$\Delta V = V \beta \Delta T = (125 \text{ cm}^3)(69 \times 10^{-6} / ^\circ \text{C})(50.0^\circ \text{C}) = 4.3 \times 10^{-1} \text{ cm}^3 = 0.43 \text{ cm}^3$$

The volume of the cube increases by 0.43 cm^3

7.2.1 Specific Heat and Latent Heat

Specific Heats:

Heat flowing into or out of a body (or system) changes the temperature of the body (or system) except during phase changes the temperature remains constant. The quantity of heat, Q , required to change the temperature of a body of mass m by is proportional to both the mass and the change in temperature. Mathematically,

$$Q \sim m \Delta T \Rightarrow Q = mc \Delta T$$

c is a proportionality constant called specific heat capacity (or in short specific heat) of the substance defined as the amount of heat required to raise the temperature of a unit mass of any substance through a unit degree. Its SI unit is J/kg.K or $\text{J/kg.}^\circ \text{C}$. The amount of heat

7.2. THE EFFECT OF HEAT ON MATERIALS (CHANGE OF TEMPERATURE, EXPANSION, C

required to change the temperature of n moles of a substance, usually for gases, by ΔT is :

$$Q = nC\Delta T$$

where C is heat capacity.

The heat capacity(C) is defined as the amount of heat energy required to raise the temperature of a substance by $1^{\circ}C$.

Example

A 0.500 kg aluminum pan on a stove is used to heat 0.250 liters of water from $20.0^{\circ}C$ to $80.0^{\circ}C$.

(a) How much heat is required? What percentage of the heat is used to raise the temperature of

(b) the pan and (c) the water?

Solution

Because water is in thermal contact with the aluminum, the pan and the water are at the same temperature. Calculate the temperature difference:

$$\Delta T = T_f - T_i = 60.0^{\circ}C$$

. Calculate the mass of water. Because the density of water is 1000 kg/m^3 , one liter of water has a mass of 1 kg, and the mass of 0.250 liters of water is $m_w = 0.250 \text{ kg}$. Calculate the heat transferred to the water. Use the specific heat of water $4186 \text{ J/kg}^{\circ}C$:

$$Q_w = mwc_w\Delta T = (0.250 \text{ kg})(4186 \text{ J/kg}^{\circ}C)(60.0^{\circ}C) = 62.8 \text{ kJ}$$

. Calculate the heat transferred to the aluminum. Use the specific heat for aluminum in Table 1:

$$Q_{Al} = m_{Al}c_{Al}\Delta T = (0.500 \text{ kg})(900 \text{ J/kg}^{\circ}C)(60.0^{\circ}C) = 27.0 \times 10^4 \text{ J} = 27.0 \text{ kJ}.$$

Compare the percentage of heat going into the pan versus that going into the water. First, find the total transferred heat:

$$Q_{Total} = Q_w + Q_{Al} = 62.8 \text{ kJ} + 27.0 \text{ kJ} = 89.8 \text{ kJ}$$

. Thus, the amount of heat going into heating the pan is

$$27.0 \text{ kJ} / 89.8 \text{ kJ} \times 100\% = 30.1\%$$

and the amount going into heating the water is

$$62.8 \text{ kJ} / 89.8 \text{ kJ} \times 100\% = 69.9\%$$

Latent Heats

Latent Heat the heat required per unit mass of a substance to produce a phase change at constant temperature. The latent heat, Q_L required to change the phase of m mass of a body at constant temperature is calculated as,

$$Q_L = \pm mL$$

Where L is the specific latent heat required to change the phase of 1 kg of a substance at constant temperature.

Types of Latent Heat Transfer

There are two types of latent heat transfers between an object and its environment.

Latent Heat of Fusion (L_f): is the heat absorbed or released when matter melts, changing phase from solid to liquid form at constant temperature. For example, 333.7 kJ of heat is required to change 1 kg of ice to water at 0°C , so for water $L_f = 333.7 \text{ kJ/kg}$.

Latent Heat of Vaporization (L_V): is the heat absorbed or released when matter vaporizes, changing phase from liquid to gas phase at constant temperature. To change 1 kg of water to steam at 100°C , 2256 kJ of heat is required and so $L_V = 2256 \text{ kJ/kg}$.

Example

1. If the amount of heat needed for a phase change is 300 kcal, calculate the latent heat of a 5 kg material.

Solution:

Given parameters are, $Q = 300 \text{ k.cal}$ $M = 5 \text{ kg}$ The formula for latent heat is given by,

$$L = Q/M = 300/5 = 60 \text{ k.cal/kg}$$

Hence latent heat value is 60 k.cal/kg

2. At 20°C , a piece of metal has a density of 60g. When immersed in a steam current at

100°C , 0.5g of the steam condenses on it. Provided that the latent heat of steam is 540 cal/g, calculate the specific heat of the metal.

Solution:

Let c be the specific heat of the metal.

Heat gained by the metal

$$Q = mc\Delta t \Rightarrow Q = 60 \times c \times (100 - 20) \Rightarrow Q = 60 \times c \times 80\text{cal}$$

The heat released by the steam

$$Q = m \times LQ = 0.5 \times 540\text{cal}$$

By the principle of mixtures, Heat given is equal to Heat taken

$$0.5 \times 540 = 60 \times c \times 80 \Rightarrow c = 0.056\text{cal/g}^{\circ}\text{C}$$

Hence specific heat value is

$$0.056\text{cal/g}^{\circ}\text{C}$$

Hence, we can conclude that the specific latent heat (L) of a material:

7.3 Laws of thermodynamics (zeros, first and second Laws)

7.3.1 Zeros Laws of thermodynamics

The zeroth law of thermodynamics states that: “Two bodies that are separately in thermal equilibrium with a third body must be in thermal equilibrium with each other.” When two bodies are in thermal equilibrium then there is no net transfer of energy between them. From our everyday experience, the zeroth law may seem obvious, but it provides us with a way of defining temperature: it is the property of a body that determines whether it is in thermal equilibrium with other bodies. This also enables accurate calibration between thermometers of different kinds.

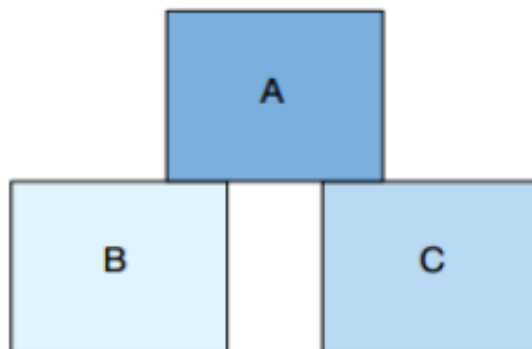


Figure 7.1: *If A is in thermal equilibrium with B, and C is in thermal equilibrium with B, then A is also in thermal equilibrium with C.*

7.3.2 First Laws of thermodynamics

The work of Joule mentioned at the start of this section led to the idea that energy as a quantity is conserved whenever any process takes place. This notion is expressed most often as the ‘law of conservation of energy’, which is a simplification of the first law of thermodynamics. The first law states that: “The increase in internal energy of a system is equal to the sum of the energy entering the system through heating, and the work done on the system.” When defining the three quantities, particular attention must be paid to the sign of each quantity. These have the following definitions:

ΔU = increase in internal energy of the system

ΔQ = the amount of energy transferred to the system by heating it (that is, by means of a temperature gradient)

ΔW = the amount of work done on the system

The first law of thermodynamics is therefore written as:

$$\Delta U = \Delta Q + \Delta W \quad (7.7)$$

Isochoric process

In a constant volume process, the volume of the system stays constant. Consequently, $W = 0$. From the first law we see that, All the heat entering the system goes into increasing the internal energy.

Adiabatic Process

In an adiabatic process, the system does not exchange heat with its surroundings; that is, $Q = 0$. The first law for an adiabatic process takes the form

$$\Delta U = W$$

Isothermal Process

It is a process which involves no change in the temperature of the system. If the process occurs at constant temperature then there is no change in the internal energy of the system so $\Delta U = 0$. The first law for an isothermal process takes the form

$$\Delta U = Q + W$$

$$0 = Q + W$$

$$Q = -W$$

Isobaric process

In an isobaric process the expansion or compression occurs at constant pressure. Any work done by the system will result in an increase in volume. The work done in Pressure- Volume graph is equal to the area under the PV graph. For an isobaric process the work done W is calculated as

$$W = P\Delta V = P(V_f - V_i)$$

The first law for an isobaric process can be written as

$$\Delta U = Q + W \text{ or } Q - P\Delta V = Q - P(V_f - V_i)$$

Entropy and the second law of thermodynamics

The second law of thermodynamics states that: "No process is possible in which there is an overall decrease in the entropy of the universe."

Review of unit questions

Table 7.3: Summary of some thermodynamic processes

Process	Meaning
Adiabatic	No heat transfer (heating or cooling result from pressure change - Work is either done on or by the gas)
Isobaric	Constant volume (also called isometric)
Isothermal	Constant Temperature
Isobaric	Constant Pressure

1. Explain what is meant by internal energy. Hence suggest how the internal energy of a real gas differs from that of an ideal gas.
2. A heat engine operating between 100°C and 700°C has efficiency equal to 40% of the maximum theoretical efficiency. How much energy does this engine extract from the hot reservoir in order to do 5000 J of mechanical work?

Short reference material

Chapter 8

Electrostatics

Learning objectives

At the end of the unit, students will be able to

- State Coulomb's law and solve problems based on it
- Define an electric field and calculate it due to point charges,
- Distinguish between the direction of the Electric Field of positive and negative charges
- Draw Electric Field Lines
- Discuss the Electrostatics field of the conductor
- Define electric potential and electric potential energy
- Derive an expression for the potential at a point p at a distance r from the charge
- Find the potential difference between the two points
- calculate capacitance

8.1 Coulomb's law

Coulomb's law gives a relation between two charges Q_1 and Q_2 which are at a separation r apart. Experiments show that the forces between two bodies obey an inverse square law and that the Force is proportional to the product of the charges. Simply, Coulomb's law states

The Force between two charges at a distance, r apart, is directly proportional to the product of the two charges and inversely proportional to the square of the distance between them. Mathematically this is written as

$$F = K \frac{Q_1 Q_2}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2}$$

Where $K = \frac{1}{4\pi\epsilon_0} = 9.0 \times 10^9 \text{ Nm}^2/\text{C}^2$ is a constant and is the permittivity of free space. ϵ_0 is a constant called the permittivity of free space (or vacuum permittivity). It has a value of $8.85 \times 10^{-12} \text{ F/m}$. This constant is fundamental to the study of electric fields. It links electrical concepts such as electric charge to mechanical quantities such as length.

Along with the permittivity of free space, there is a similar constant relating to magnetic fields. This is called the permeability of free space (μ_0).

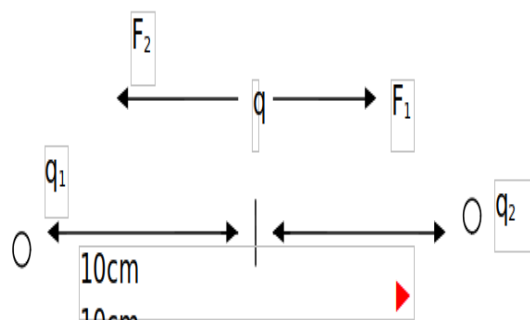
The Force between similar charges is repulsive, and the Force between unlike charges is attractive. In the case of gravitational Force, we can have only attractive Force due to masses. When two charges exert forces simultaneously

on a third charge, the total Force acting on that charge is the vector sum of the forces that the two charges would exert individually. This important property, called the principle of superposing, holds for any number of charges.

$$F = F_1 + F_2 + F_3 + \dots + F_n$$

Example

A test charge of $q = +1 \times 10^{-6} \text{ C}$ is placed halfway between a charge of $q_1 = +5 \times 10^{-6} \text{ C}$ and a charge of $q_2 = +3 \times 10^{-6} \text{ C}$ that are 20cm apart in the figure below. Find the magnitude and direction of the Force on the test charge **Solution**



The Force exerted on the test charge

$$F_1 = K \frac{qq_1}{r_1^2} = (9 \times 10^9 Nm^2/c^2) \frac{(-1 \times 10^{-6}c)(+5 \times 10^{-6}c)}{(0.1)^2} = +4.5N$$

This Force is to the right and taken as positive. The Force exerted by the charge q_2 on q is

$$F_2 = K \frac{qq_2}{r_2^2} = (9 \times 10^9 Nm^2/c^2) \frac{(1 \times 10^{-6}c)(+3 \times 10^{-6}c)}{(0.1)^2} = +2.7N$$

This Force is to the left. If the right is taken as positive, F_2 is taken as negative

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = 4.5N - 2.7N = 1.8N$$

and it acts to the right, that is, towards the $+3 \times 10^{-6}c$ charge

Exercise

1. Two charges, one of $+5 \times 10^{-7}c$ and the other $-2 \times 10^{-7}c$ attract each other with a force of 100N. How far apart are they?
2. 3c and 5c charges are separated by 2m. Where between these charges is a third charge placed, in order for the net Force on it to be zero?
3. Three identical charges of $2\mu c$ are placed at (-3, 0) m, (3, 0) m, and (0, 4) m in a rectangular coordinate system. What is the resultant Force on the charge that is placed at (0, 4) m

8.2 Electric Field (E)

The concept of an electric field is used to visualize how a charge, or a collection of charges, influences the region around it. The electric field E is analogous to g , which we call the acceleration due to gravity, but which is the gravitational field. Everything we learned about gravity, and how masses respond to gravitational forces can help us understand how electric charges respond to electric forces.

The electric field concept arose to explain action-at-a-distance forces. All charged objects create an electric field that extends outward into the surrounding space. The charge alters that space, causing any other charged thing that enters the space to be affected by this field. The strength of the electric field is dependent upon how charged the object creating the field and upon the distance of separation from the charged objects

8.2.1 Electric Field Intensity

Electric field strength is a vector quantity; it has both magnitude and direction. The magnitude of the electric field strength is defined in terms of how it is measured. Let's suppose that an electric charge can be denoted by the symbol Q . This electric charge creates an electric field; since Q is the source of the electric field, we will refer to it as the source charge. The strength of the source charge's electric field could be measured by any other charge placed somewhere in its surroundings. The charge used to measure the electric field strength is referred to as a test charge since it is used to test the field strength. The test charge has a quantity of charge denoted by the symbol q . When placed within the electric field, the test charge will experience an electric force that is either attractive or repulsive. As is usually the case, this Force will be denoted by the symbol F . The electric field's magnitude is defined as the Force per charge on the test charge.

$$\text{Electric Field} = \frac{\text{Force}}{\text{Charge}}$$

If the symbol E denotes the electric field strength, then the equation can be rewritten in symbolic form as

$$\vec{E} = \frac{\vec{F}}{q} \hat{r}$$

Where $\hat{r}$ is a unit vector

The Electric field $\vec{E}$ at a point in space is defined as the electric force $\vec{F}$ acting on a positive test charge q placed at the point divided by the test charge

The standard metric units of electric field strength arise from its definition. Since the electric field is defined as a force per charge, its units would be force units divided by charge units. In this case, the standard metric unit is Newton/Coulomb (N/C)

The electric field strength is not dependent upon the quantity of the test charge. Now we will investigate a new equation that defines electric field strength in terms of the variables which affect the electric field strength. To do so, we will have to revisit the Coulomb's Law equation. When applied to our two charges - the source charge (Q) and the test charge (q).

$$\vec{F} = K \frac{qQ}{r^2} \hat{r}$$

The formula for electric Force can be written as A new equation can be derived if the expression for electric Force given by Coulomb's law is substituted for Force in the above $E = F/q$

equation.

$$\vec{E} = K \frac{Q}{r^2} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

The electric field strength depends upon the quantity of charge on the source charge Q and the distance of separation r from the source charge.

The strength of an electric field created by source charge Q is inversely related to the square of the distance from the source. This is known as an inverse square law.

Electric field strength is location dependent, and its magnitude decreases as the distance from a location to the source increases. By whatever factor the distance changes, the electric field strength will change inversely by the square of that factor

If a number of point charges $Q_1, Q_2, Q_3, \dots, Q_n$ are at a distance $r_1, r_2, r_3, \dots, r_n$ from a given point P. each exerts a force on a test charge q placed, and the resultant Force on the test charge is the vector of some of these forces.

$$\vec{E} = E_1 + E_2 + E_3 + \dots + E_n = \sum_{i=1}^n \vec{E}_i$$

Because each term to be summed is a vector, the sum is a vector sum. The fact that the fields that would be caused by the individuals charge is a direct result of the principle of super position.

Activity 8 .1

Explain what happens to the magnitude of the electric field created by a point charge as r approaches zero

Activity 8 .2

Consider two equal positive or negative point charges separated by the distance d . At what point (other than) would a third test charge experience no net force?

Example

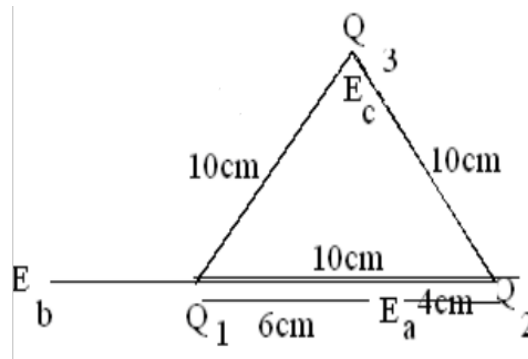
- 1 What are the magnitude and direction of the electric field 1.5cm from a fixed point charge of $+1.2 \times 10^{-10}C$? **Solution** The magnitude of the electric field is computed from

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = K \frac{Q}{r^2} \hat{r} = 9 \times 10^9 Nm^2/c^2 \times \frac{+1.2 \times 10^{-10}C}{(0.015m)^2}$$

$$\vec{E} = 4.8 \times 10^3 N/c$$

- Notice that r was expressed in the SI unit of meters
- The direction of the field is outward from the point charge because the charge is positive

1 Point charge Q_1 and Q_2 of $12 \times 10^{-9}C$ and $-12 \times 10^{-9}C$ respectively, are placed 0.1m apart as shown Compute the electric fields due to the charge at point a, b, and c



Solution: At point a, the vector due to the positive charge Q , is directed toward the right, And the magnitude is

$$\vec{E}_1 = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = 9 \times 10^9 Nm^2/c^2 \times \frac{12 \times 10^{-9}C}{(0.06m)^2} = 3 \times 10^4 N/c \text{ to the right}$$

The vector due to the negative charge Q_2 is toward the right, And the magnitude is

$$\vec{E}_2 = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = 9 \times 10^9 Nm^2/c^2 \times \frac{12 \times 10^{-9}C}{(0.04m)^2} = 6.75 \times 10^4 N/c \text{ to the right to the right}$$

Hence at point a

$$\vec{E} = (3 + 6.75) \times 10^4 N/c = 9.75 \times 10^4 N/c$$

At point b

The vector due to q_1 , is directed toward the left, with magnitude

$$\vec{E}_1 = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = 9 \times 10^9 Nm^2/c^2 \times \frac{12 \times 10^{-9}C}{(0.04m)^2} = 6.75 \times 10^4 N/c \text{ to the right}$$

The vector due to the negative charge Q_2 is toward the right, And the magnitude is

$$\vec{E}_2 = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = 9 \times 10^9 Nm^2/c^2 \times \frac{12 \times 10^{-9}C}{(0.14m)^2} = 0.55 \times 10^4 N/c \text{ to the right to the left}$$

Hence at point b

$$\vec{E} = (6.75 - 0.55) \times 10^4 N/c = 9.75 \times 10^4 N/c \text{ to the left}$$

At point C, the magnitude of each vector is

$$\begin{aligned}\vec{E} &= \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} = 9 \times 10^9 Nm^2/c^2 \times \frac{12 \times 10^{-9} C}{(0.10m)^2} = 1.08 \times 10^4 N/c \\ \vec{E}_x &= \left(K \frac{Q_1}{r^2} \cos(60^\circ) - K \frac{Q_2}{r^2} \cos(60^\circ) \right) \\ &= \frac{2KQ}{r^2} \cos(60^\circ) \\ &= \frac{2 \times 9 \times 10^9 \times 12 \times 10^{-9}}{(0.1)^2} \times 0.5 N/c = 1.08 \times 10^4 N/c \text{ to the right} \\ \vec{E}_y &= (E_1)_y + (E_2)_y = 0\end{aligned}$$

$$\vec{E} = E_a + E_b + E_c = 9.75 \times 10^4 N/c - 6.20 \times 10^4 N/c + 1.08 N/c = 4.63 \times 10^4 N/c \text{ to the right}$$

8.3 Electric Field Lines

A more useful means of visually representing the vector nature of an electric field is through the use of electric field lines of Force. These patterns of lines, sometimes referred to as electric field lines, point in the direction which a positive test charge would accelerate if placed upon the line. As such, the lines are directed away from positively charged source charges, and toward negatively charged source charges.

The electric field can be represented graphically by field lines. These lines are drawn in such a way that, at a given point, the tangent of the line has the direction of the electric field at that point. The density of lines is proportional to the magnitude of the electric field. Each field line starts on a positive point charge and ends on a negative point charge. Since the density of field lines is proportional to the strength of the electric field, the number of lines emerging from a positive charge must also be proportional to the charge.

Electric field lines provide a means to visualize the electric field. Since the electric field is a vector, electric field lines have arrows showing the direction of the electric field. Lines of Force are also called field lines. The direction of the field line at a point tells you what direction the Force experienced by a charge will be if the charge is placed at that point. If the

charge is positive, it will experience a force in the same direction as the field; if it is negative the Force will be opposite to the field. The density of lines surrounding any given source

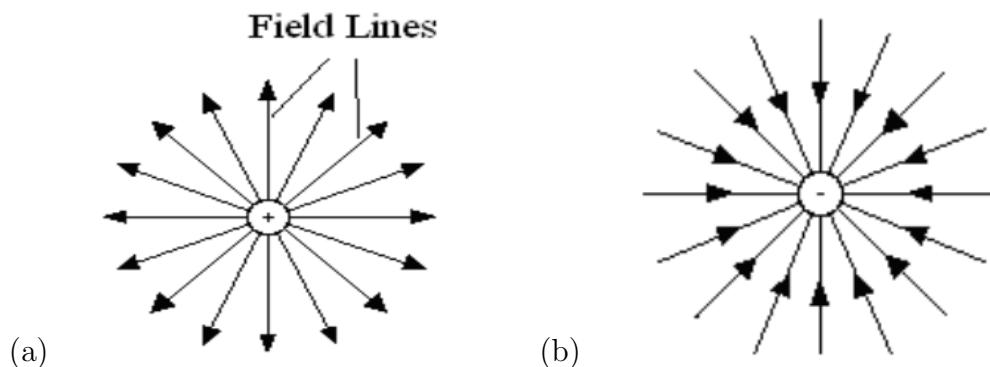


Figure 8.1: *Electric field from an isolated, (a) Positive Charge (b) Negative Charge*

charge is proportional to the quantity of a charge on that source charge. If the quantity of charge on a source charge is not identical, the pattern will take on an asymmetric nature as one of the source charges will have a greater ability to alter the electrical nature of the surrounding space.

There are a number of principles which will assist in such predictions. These principles are

- Electric field lines always extend from a positively charged object to a negatively charged object, from a positively charged object to infinity, or from infinity to a negatively charged object
- Electric field lines never cross each other
- Electric field lines are most dense around objects with the greatest amount of charge.
- At locations where electric field lines meet the surface of an object, the lines are perpendicular to the surface.

Activity 8.3

A charge $4q$ is at a distance r from a charge $-q$. Compare the number of electric field Lines leaving the charge $4q$ with the number entering the charge $-q$ where do the extra lines beginning on $4q$ end.

Activity 8.4

A test charge is released in the field due to two point charges. Do the field lines indicate the possible path traveled by the test charge?

Conductors in Electrostatic Fields

In general a conductor can be defined as a region in space where charges are free to move (e.g. a metal). In a static situation the charges don't move. This implies that there is no field within the conductor. Thus, inside a conductor:

$$E(r) = 0$$

A conductor is in electrostatic equilibrium when the charge distribution (the way the charge is distributed over the conductor) is fixed. Basically, when you charge a conductor the charge spreads itself out. At equilibrium, the charge and electric field follow these guidelines:

- the excess charge lies only at the surface of the conductor
- the electric field is zero within the solid part of the conductor
- the electric field at the surface of the conductor is perpendicular to the surface
- charge accumulates, and the field is strongest, on pointy parts of the conductor

8.4 Electric potential of a point charge

The electric potential of a point charge is

$$V = kQ/r$$

. where k is a constant equal to $9.0 \times 10^9 \text{ Nm}^2/\text{C}^2$. Electric field is a vector while electric potential is a scalar. The voltage resulting from a combination of point charges is obtained by adding voltages as integers, whereas the overall electric field is obtained by adding individual fields as vectors.

Point charges, such as electrons, are among the fundamental building blocks of matter. Furthermore, spherical charge distributions (like on a metal sphere) create external electric fields exactly like a point charge. The electric potential due to a point charge is, thus, a case we need to consider. Using calculus to find the work needed to move a test charge q from a large distance away to a distance of r from a point charge Q , and noting the connection between work and potential

$$W = -q\Delta V$$

, we can define the electric potential V of a point charge:

Electric potential

Consider a charge q placed in an electric field E . Let us choose some arbitrary reference point A in the field at this point the electric potential energy of the charge is defined to be zero. This defines the electric potential energy of the charge at every other point in the field. For instance, the electric potential energy U_B at some point B is simply the work W done in moving the charge from A to B along any path: It is clear that this depends on both the particular charge q which we place in the field and the magnitude and direction of the electric field along some arbitrary route between points A and B . We can exploit this fact to define a quantity known as the electric potential. The difference in electric potential between two points B and A in an electric field is simply the work done in moving some charge between the two points divided by the magnitude of the charge. Thus,

$$V_B - V_A = \frac{\Delta W}{q} = \frac{\Delta U}{q}$$

The general expression for the electrical potential of a point charge Q can be obtained by referencing to a zero of potential at infinity. The expression for the potential difference then.



r_B goes to infinity is given simply

$$V = \frac{KQ}{r} = \frac{Q}{4\pi\epsilon_0 r}$$

The zero of electric potential (voltage) is set for convenience, but there is usually some physical or geometric logic to the choice of the zero point. For a single point charge or localized collection of charges, it is logical to set the zero point at infinity. If there are a number of charges in space, the potential at a point is found by superposition principles that is the electric potential due to a number of charges is the algebraic sum of the individuals

potentials. The total electric potential point p is the sum of the potential due to charges

$$Q_1, Q_2, Q_3, \dots, Q_n$$

$$V = \frac{KQ_1}{r_1} + \frac{KQ_2}{r_2} + \dots + \frac{KQ_n}{r_n} = \sum_{i=1}^n \frac{Q_i}{r_i}$$

Note; potential a scalar quantity

Activity 8.5

In a certain region of space the electric field is zero From this we can conclude that the electric potential in this region is A) zero B) constant C) positive D) negative

The dimensions of electric potential are work (or energy) per unit charge. The units of electric potential are, therefore, joules per Coulomb (J/c). A joule per Coulomb is usually referred to as a volt (V)

$$1J/c = 1V$$

Consider a charge q which is slowly moved a small distance +x along the x-axis. Suppose that the difference between the electric potential at the final and initial positions of the charge is +V. By definition, the change +U in the charge's electric potential energy is given by

$$+U = q + V = W$$

$$q\Delta V = qE\Delta r$$

$$\Delta V = E\Delta r$$

$$E = \frac{\Delta V}{\Delta r}$$

Where E is the electric field strength According to equation electric field strength has the dimension of potential difference over the length. It follows that the unit of electric field is volt per mete (Vm).

8.4.1 Motion of charged particles in an electric field

When a particle of charge of and mass m is placed in an electric field E, the electric Force exerted on the charge is qE. If this is the only Force exerted on the particle it must be the net Force and cause the particle to accelerated according to Newton's second law

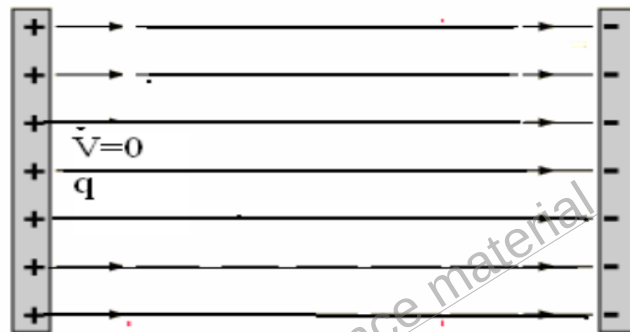
$$F_e = qE = am$$

$$a = \frac{qE}{m}$$

If E is uniform the acceleration is constant. If the particle has a positive charge, its acceleration is in the direction of the electric field. If the particle has a negative charge, its acceleration is in the direction of opposite the electric field.

Example

1. As shown in the figure below, a positive point charge q of mass m is released from rest in a uniform electric field E directed along the x -axis. Describe its motion



Solution

The acceleration a is constant and is given by qE/m . The motion is simple linear motion along the x axis. Therefore we can apply the equation of kinematics in one dimension

$$x_f = x_i + v_i t + \frac{1}{2} a t^2$$

$$v_f = v_i + a t$$

$$v_f^2 = v_i^2 + 2a(x_f - x_i)$$

Choosing the initial position of the charge as $x_i = 0$ and assigning $V_i = 0$ because the particle starts from rest, the position of the particle as a function of time is

$$x_f = \frac{1}{2} a t^2 = \frac{qE}{m} t^2$$

The speed of the particle is given by

$$v_f = \frac{qE}{m} t$$

The third kinematics equation gives us

$$v_f^2 = 2ax_f = \left(\frac{2qE}{m}\right)x_f$$

from which we can find the kinetic energy of the charge after it has moved a distance

$$KE = \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(\frac{2qE}{m}\right)(x_f - x_i) = qE\Delta x$$

2 An electron enters the region of a uniform electric field as shown with $V_i = 3 \times 10^6$ m/s and $E = 200$ N/C. The horizontal length of the plate, $L = 0.100$ m

- i) Find the acceleration of the electron while it is in the electric field
- ii) If the electron enters the field at $t = 0$ find the time at which it leaves the field
- iii) If the vertical position of the electron as it enters the field is $y_i = 0$. What is the vertical position when it leaves the field?

Solution

i) The charge on the electron has $1.6 \times 10^{-19} C$ and $m_e = 9.11 \times 10^{-31} kg$

$$a = -\frac{qE}{m_e} = \left(\frac{1.6 \times 10^{-19} C}{9.11 \times 10^{-31} kg}\right)$$

$$a = -3.51 \times 10^{13} m/s^2$$

ii) The horizontal distance across the field is $L = 0.1$ m. We find that the time at which the electron exists the electric field is

$$L = 0.1 m$$

$$t = \frac{L}{v_i} = \frac{0.1 m}{3 \times 10^6 m/s} = 3.33 \times 10^{-8} s$$

iii) Using the results from part A and B we find that

$$y_f = \frac{1}{2}a_y t^2 = -\frac{1}{2}(3.51 \times 10^{13} m/s^2)(3.33 \times 10^{-8} s)^2 = -0.0195 m = -1.95 cm$$

8.5 Capacitance and Capacitor networks

A capacitor is a device that is used to store electric charge. It is usually made up of two plates separated by a thin insulating material known as the dielectric. The capacitance of a system depends only on its shape and on the insulators it contains. One plate of the capacitor is positively charged, while the other has a negative charge. The charge in a capacitor is

proportional to the potential difference between the plates. For a capacitor with charge Q on the positive plate and $-Q$ on the negative plate, the capacitance measures the amount of charge a capacitor can store. A convenient measure of the ability of a device to store electric charge is its capacitance C .

A battery will transport charge from one plate to another until the voltage produced by the charge buildup is equal to the battery voltage. The capacitance of an object is defined as being equal to the charge required to raise the potential of that object by one V

$$C = \frac{Q}{V}$$

Or

$$Q = CV$$

Where C is the capacitance in Farad Q is the charge in Coulomb stored in each plate V is the potential in Volts applied to the plate

The capacitor's capacitance (C) is a measure of the amount of charge (Q) stored on each plate for a given potential difference or voltage (V) which appears between the plates.

The SI unit of capacitance is the farad F The circuit symbol for a capacitor is $1\text{farad}=1\text{Coulomb}/1\text{Volt}$

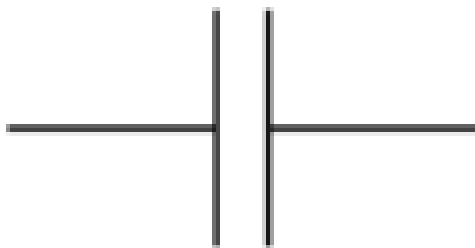


Figure 8.2: *The circuit symbol for a capacitor*

Activity

A $25\mu F$ capacitor is charged to a potential of $18V$. How much charge stored on capacitor?

8.6 The Parallel Plate Capacitor

Consider two large flat plates placed near one another. The plates are parallel, and have equal and opposite charges uniformly distributed. This configuration is known as a parallel-plate capacitor. A parallel-plate capacitor is a great way to create a uniform field.

Consider a capacitance C in vacuum consisting of two parallel plates, each with area A separated by a distance d as shown in figure 1. One plate carries a charge Q , and the other carries a charge $-Q$. The amount of a charge that can be stored on a plate for a given potential increases as a plate area increased. Thus, we expect the capacitance to be proportional to the plate area A .

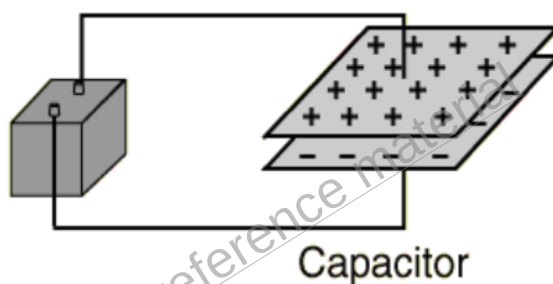


Figure 8.3: *The Parallel Plate Capacitor*

Now consider the region that separates the plate. The electric field between the plates must increase as d decreased. Moving the plates together causes the charge on the capacitor to increase. If d is increased, the charge decreases. As a result, we expect the capacitance of the pair of the plates to be inversely proportional to d . We can verify these physical arguments with the following derivation. The surface charge density on either plate is

$$\delta = \frac{Q}{A}$$

The magnitude of the Electric field has a very simple relation to the voltage between the plates and their separation d .

$$E = \frac{V}{d}$$

Using the definition of capacitance we can determine the capacitance C of an ideal capacitor as a function of its structure.

$$\begin{aligned} C &= \frac{Q}{V} = \frac{Q}{Ed} = \frac{Q}{\frac{\delta}{\epsilon_0}d} \\ &= \epsilon_0 \frac{Q}{\frac{\delta}{A}d} \\ C &= \epsilon_0 \frac{A}{d} \end{aligned}$$

This equation for the capacitance of a parallel capacitor shows that C is a constant independent of the charge stored in on the plates or the voltage across the capacitor. The capacitance of a system depends on its shapes, dimensions and separation of the conductors that make up the capacitor

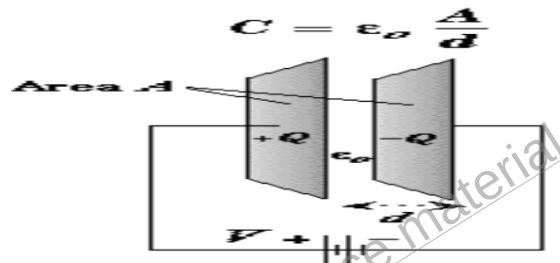


Figure 8.4: The Parallel Plate Capacitor

Example

- 1) The plates of a parallel-plate capacitor are 5mm apart and 2m² in area. The plates are in vacuum. A potential difference 10, 000v is applied across the capacitor compute
 - A) The capacitance
 - B) The charge on the plate
 - C) The electric intensity in the space b/n them

Solution A)

$$\begin{aligned} C &= \epsilon_0 \frac{A}{d} = \left(\frac{8.85 \times 10^{-12} \text{C}^2/\text{Nm}^2}{5 \times 10^{-3} \text{m}} \right) (2 \text{m}^2) \\ C &= 3.54 \times 10^{-9} \text{C}^2/\text{Nm} = 3.54 \times 10^{-9} \text{F} \end{aligned}$$

B) The charge on the plate

$$Q = CV_{ab} = (3.54 \times 10^{-9} \text{C/V})(10^4 \text{V}) = 3.54 \times 10^{-5} \text{C}$$

C) The electric intensity in the space b/n them

$$E = \frac{\delta}{d} = \frac{Q}{\epsilon_0 A} = \frac{3.54 \times 10^5 C}{8.85 \times 10^{-12} C^2/Nm(2m^2)}$$

$$= 20 \times 10^5 N/c$$

Since the electric field equals the potential gradient

$$E = \frac{V_{ab}}{d} = \frac{10^4 V}{5 \times 10^{-3} m} = 20 \times 10^5 v/m$$

$$\frac{N}{C} = \frac{V}{m}$$

2 Parallel-plate capacitor is designed to have a capacitance of 1.00F when the plates are separated by 1.00mm in vacuum what must be the arch of the plates. (Ans $A = 1.13 \times 10^8 m^2$)

8.6.1 Energy Stored in a Capacitor

The energy stored in a capacitor is the same as the work needed to build up the charge on the plates. As the charge increases, the harder it is to add more. Potential energy is the charge multiplied by the potential, and as the charge builds up the potential does too. If the potential difference between the two plates is V at the end of the process, and 0 (zero) at the start, the average potential is V/2. Multiplying this average potential by the charge gives the potential energy.

$$PE = 1/2 QV$$

. Substituting in for Q, $Q = CV$, gives: The energy stored in a capacitor is: intermes of C and V.

$$W = U = \frac{1}{2} CV^2$$

Substituting $Q = CV$ and $V = \frac{Q}{C}$

$$U = \frac{1}{2} QV$$

This is U intermes of Q and V

$$U = \frac{Q^2}{2C}$$

intermes of Q and C Where U=Electric potential energy in joule Q=Charge in Coulomb. V=Potential in volt C = Capacitance in farad These formulae are valid for any type of capacitor, since the arguments we used to derive them do not depend on any special property

of parallel plate capacitors. The potential difference between the plates is $V = Ed$ and $C = \frac{\epsilon_0 A}{d}$. Thus, the energy stored in the capacitor can be written as

$$U = W = \frac{CV^2}{2}$$

$$V = Ed, C = \frac{\epsilon_0 A}{d}$$

$$U = \frac{\epsilon_0 A E^2 d^2}{2d}$$

$$U = \frac{\epsilon_0 A E^2 d}{2}$$

Now, Ad is the volume of the field filled region between the plates, so if the energy is stored in the electric field then the energy per unit volume, or energy density, of the field must be

$$u = \frac{U}{V} = \frac{1}{2} \frac{CV^2}{Ad}$$

Substituting

$$C = \epsilon_0 \frac{A}{D}$$

$$\frac{V^2}{d^2} = E^2$$

$$u = \frac{1}{2} \epsilon_0 \frac{V^2}{d^2} = \frac{1}{2} \epsilon_0 E^2$$

- 1) Air filled parallel plate capacitor has a capacitance of 5.0 pF. A potential of 100V is applied across the plates, which are 1.0 cm apart, using a storage battery.
 - a) What is the energy stored in the capacitor? Suppose that the battery is disconnected and the plates are moved until they are 2.0 cm apart.
 - b) What is the energy stored in the capacitor now?
 - c) Suppose, instead, that the battery is left connected and the plates are again moved until they are 2.0 cm apart. What is the energy stored in the capacitor in this case?

Solution

The initial energy stored in the capacitor is

$$U = \frac{CV^2}{2} = \frac{5 \times 10^{-12}}{2} (1000)^2 J = 2.58 \times 10^{-8} J$$

When the spacing between the plates is doubled, the capacitance of the capacitor is halved to 2.5 pF. If the battery is disconnected then this process takes place at constant charge Q . Thus, it is obvious from the formula

$$U = \frac{Q^2}{2C}$$

That in this case the energy stored in the capacitor doubles. So, the new energy is

$$U = 2(2.58 \times 10^{-8})J$$

$$= 5.16 \times 10^{-8}J$$

8.7 Capacitance net work

Parallel Combination

Capacitors are one of the standard components of electronic circuits. Complicated combinations of capacitors often occur in practical circuits. It is, therefore, useful to have a set of rules for finding the equivalent capacitance of some general arrangement of capacitors. It turns out that we can always find the equivalent capacitance by repeated application of two simple rules. These rules related to capacitors connected in series and in parallel. In a parallel combination, the capacitors are usually drawn side by side. If we imagine them as parallel-plate capacitors with the same gap, snuggling them right up next to each other, the combination seems to become a single capacitor with an area equal to the sum of the areas. Then from the equation for capacitance of a parallel-plate capacitor, we have.

$$\begin{aligned} C_{eq} &= \frac{\epsilon_0 A_{eq}}{d} = \frac{\epsilon_0 (A_1 + A_2)}{d} \\ &= \frac{\epsilon_0 A_1}{d} + \frac{\epsilon_0 A_2}{d} \\ C_{eq} &= C_1 + C_2 \end{aligned}$$

Or consider two capacitors connected in parallel; i.e. with the positively charged plates connected to a common "input" wire and the negatively charged plates attached to a common "output" wire. What is the equivalent capacitance between the input and output wires? In this case, the potential difference V across the two capacitors is the same, and is equal to the potential difference between the input and output wires.

The total charge Q , however, stored in the two capacitors is di-vided between the capacitors, since it must distribute itself such that the voltage across the two is the same. Since the capacitors may have different , C_1 and C_2 , the charges Q_1 and Q_2 may also be different. The equivalent capacitance C_{eq} of the pair of capacitors is simply the ratio Q/V

$$C_{eq} = \frac{Q}{V} = \frac{Q_1}{V} + \frac{Q_2}{V}$$

$$C_{eq} = C_1 + C_2$$

When a number of capacitors are connected in parallel, the total or effective capacitance

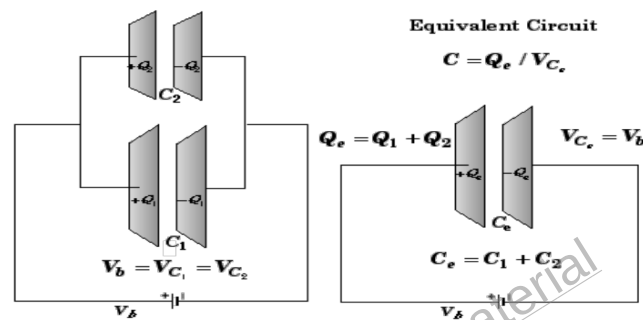


Figure 8.5: *Capacitors connected in Parallel*

of the group is equal to the sum of the individual capacitances. The equation for calculating the total capacitance C obtained by capacitances C_1, C_2, C_3 etc.. The formula for parallel capacitor is same as the resistance in series. The working voltage of parallel capacitors is equal to the lowest working voltage rating in the combination. Parallel connected Capacitors

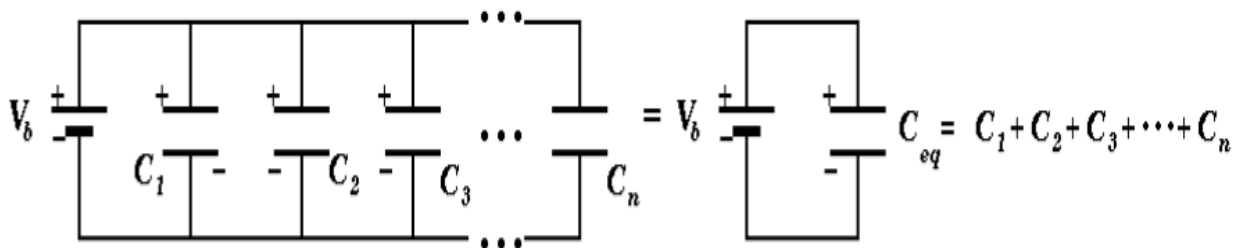


Figure 8.6: *Capacitors connected in Parallel*

always have the same voltage drop across each of them. They do not have the same charge unless they have the same capacitance C . The charge on the equivalent capacitor C_{eq} is the

sum of the charges on both capacitors. The Voltage on the equivalent capacitor C_{eq} is the same as the voltage across either capacitor.

The equivalent capacitance of capacitors connected in parallel is the sum of the individual capacitances

Series Combination

In a series combination, the capacitors are connected head-to-tail. We want to replace the pair or more by a single equivalent capacitor. To do this, we must understand how the charge is distributed on the plates.

Consider the inner pair of plates, one from each capacitor, connected by a conductor. These three objects are electrically isolated from the remainder of the circuit; they form a single isolated conductor. Since the net charge on the capacitors is zero before the battery is connected, the net charge on the inner pair of plates must also be zero. After the battery is connected, the plates of the capacitors will hold some charge, but the inner pair of plates will still have zero net charge. Therefore, the charges on the inner pair of plates are equal and opposite, and we see that both capacitors will hold the same charge. We don't add these charges together, as in the parallel case. The quantity that adds is the voltage across each capacitor. consider capacitors arranged so that the potential across the combination is equal to the sum of the potential difference across each as shown in fig

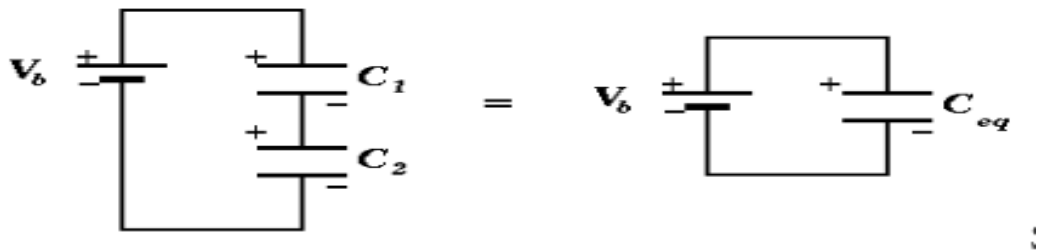


Figure 8.7: *Capacitors connected in Series*

$$\Delta V = \Delta V_1 + \Delta V_2$$

The voltage across each capacitor is related to their charges

$$\Delta V_1 = \frac{Q}{C_1}$$

and

$$\Delta V_2 = \frac{Q}{C_2}$$

The definition of equivalent capacitor is

$$C_{eq} = \frac{Q}{\Delta V}$$

Or

$$\Delta V = \frac{Q}{C_{eq}}$$

Therefore

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

For more than two Capacitor

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n} = \sum_{i=1}^n \frac{1}{C_i}$$

Series connected Capacitors always have the same charge. They do not the same voltage

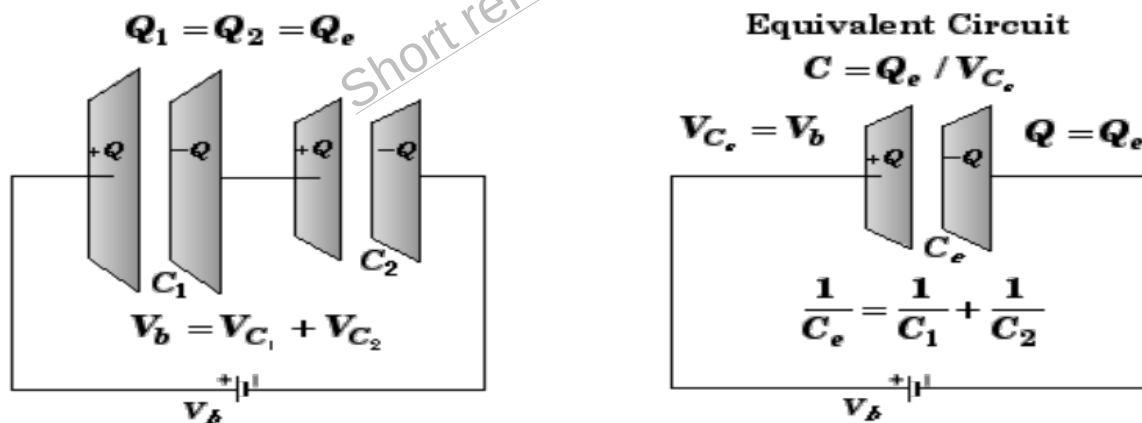


Figure 8.8: Capacitors connected in Series

unless the capacitors have the same Capacitance C . The charge on the equivalent capacitor C_e is the same as the charge on either capacitor. The Voltage across the equivalent capacitor C_{eq} is the sum of the voltage across both capacitors. If two or more capacitors are connected in series as shown above, the total capacitance is less than that of the smallest capacitor in the group

Example

- 1) Let $C_1 = 6\mu F$ and $C_2 = 3\mu F$, $V_{ab} = 18V$
- A. What is the equivalent capacitance of the series combination
- B. What is the charge on each capacitor
- C. Find the potential difference across the capacitor.

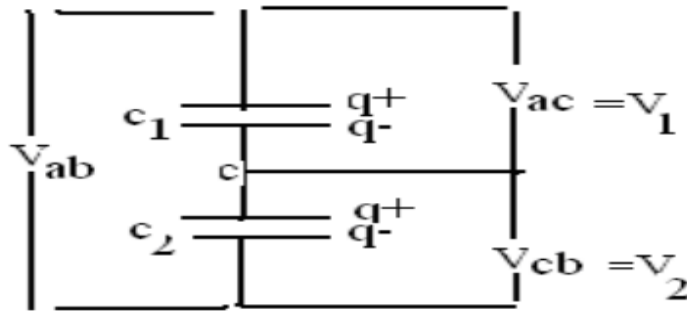


Figure 8.9: Capacitors connected in Series

Solution

a) for series combination

$$\begin{aligned}\frac{1}{C_{eq}} &= \frac{1}{C_1} + \frac{1}{C_2} \\ &= \frac{1}{6\mu F} + \frac{1}{3\mu F} = \frac{1}{2\mu F} \\ C_{eq} &= 2\mu F\end{aligned}$$

b) The charge Q is

$$Q_1 = Q_2 = C_{eq} V_{ab} = (2\mu F)(18V) = 36\mu C$$

c) The potential difference across the capacitor are

$$V_{ac} = V_1 \frac{Q}{C_1} = \frac{36\mu F}{6\mu F} = 6V$$

$$V_{cb} = V_2 = \frac{Q}{C_2} = \frac{36\mu F}{3\mu F} = 12V$$

- 1) A $1\mu F$ and a $2\mu F$ capacitor are connected in parallel and this pair of capacitors is then connected in series with a $4\mu F$ capacitor.

i) What is the equivalent capacitance of the whole combination?

- ii) What is the charge on the $4\mu\text{F}$ capacitor if the whole combination is connected across the terminals of a 6V battery?
- iii) What are the charges on the $1\mu\text{F}$ and $2\mu\text{F}$ capacitors?

Answer:

The equivalent capacitance of the $1\mu\text{F}$ and a $2\mu\text{F}$ capacitors connected in parallel is $1\mu\text{F} + 2\mu\text{F} = 3\mu\text{F}$. When a $3\mu\text{F}$ capacitor is combined in series with $4\mu\text{F}$ capacitor the equivalent capacitance of the whole combination is given by

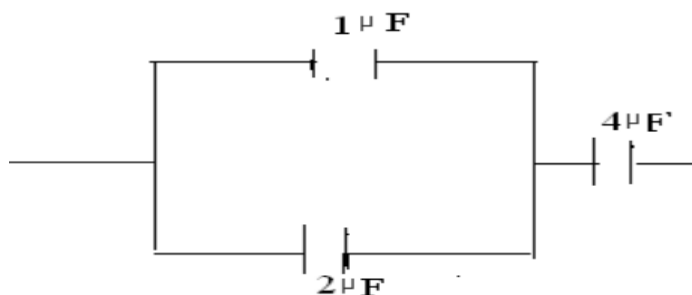


Figure 8.10: *Capacitors connected in Series*

$$\begin{aligned}\frac{1}{C_{eq}} &= \frac{1}{3\mu\text{F}} + \frac{1}{4\mu\text{F}} \\ &= \frac{7}{12 \times 10^{-6}} \text{F}^{-1} \\ C_{eq} &= \frac{12 \times 10^{-6}}{7} \text{F} = 1.71\mu\text{C}\end{aligned}$$

The charge delivered by the 6V battery is $Q = C_{eq} V = (1.71 \times 10^{-6})(6) = 10.3\mu\text{C}$. This is the charge on the $4\mu\text{F}$ capacitor, since one of the terminals of the battery is connected directly to one of the plates of this capacitor. The voltage drop across the $4\mu\text{F}$ capacitor is

$$V_A = \frac{Q}{C_4} = \frac{10 \times 10^{-6} \text{C}}{4 \times 10^{-6} \text{F}} = 2.57\text{V}$$

Thus, the voltage drop across the $1\mu\text{F}$ and $2\mu\text{F}$ combination must be

$$V_{12} = 6\text{V} - 2.57\text{V} = 3.43$$

The charge stored on the $1\mu\text{F}$ is given by

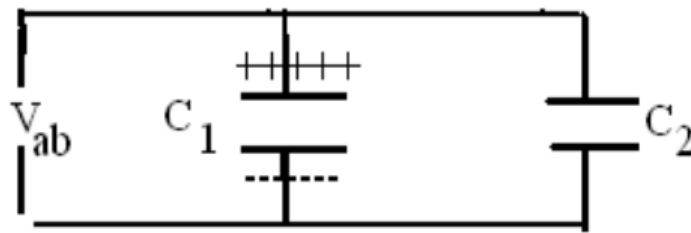
$$Q_1 = C_1 V_{12} = (1\mu\text{F})(3.43) = 3.42\mu\text{F}$$

Likewise, the charge stored on the $2\mu\text{F}$ capacitor is

$$Q_2 = C_2 V_{12} = (2\mu\text{F})(3.43) = 6.84\mu\text{F}$$

Note that the total charge stored on the $1\mu\text{F}$ and $2\mu\text{F}$ combination is

$$Q_{12} = Q_1 + Q_2 = 10.3\mu\text{C}$$



In fig $C_1 = 6\mu\text{F}$ and $C_2 = 3\mu\text{F}$ and $V_{ab} = 18\text{V}$ find

- Equivalent capacitance
- The charge on each capacitor
- The potential difference on each capacitor

Solution

A) The equivalent capacitance of the parallel combination is $C_{eq} = C_1 + C_2 = 6\mu\text{F} + 3\mu\text{F} = 9\mu\text{F}$ B) The charge Q_1 and Q_2 are

$$Q_1 = C_1 V = (6\mu\text{F})(18\text{V}) = 108\mu\text{C}$$

$$Q_2 = C_2 V = (3\mu\text{F})(18\text{V}) = 54\mu\text{C}$$

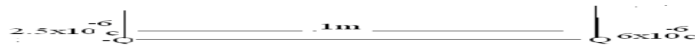
C) The potential is the same for each capacitor. Because they are connected in parallel

Solution

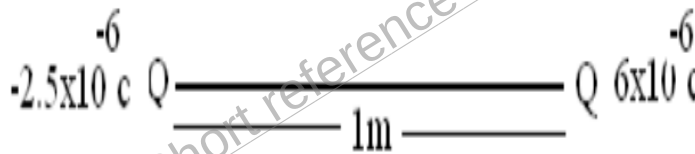
$$\begin{aligned} C_0 &= \epsilon_0 \frac{A}{d} (8.85 \times 10^{-12} \text{C}^2/\text{Nm}^2) \\ &= 17.7 \times 10^{-11} \text{F} = 177 \text{PF} \end{aligned}$$

Unit Exercise

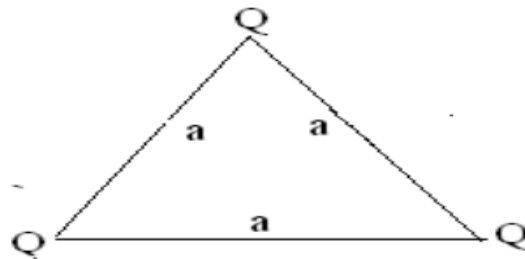
- 1) What is the magnitude and direction of the electric field that will balance the weight of?
 - a) an electron and
 - b) a proton
- 2) In figure determine the point (other than infinity) at which the electric field is zero



- 3) Two point charges are located on the X axis. The first is a charge Q at $X = -a$. The second is an unknown charge located at $X = 3a$. The net electric field these charges produce at the origin has a magnitude of $\frac{2KQ}{a^2}$. What are the two possible values of the known charge?
- 4) Determine the point at which the electric field is zero.



- 5) Three equal positive charges are at the corners of an equilateral triangle of side a as shown.



- A Three Charges together create an electric field. Sketch the field lines in the plane of the charge
- B Find the location of the point (other than infinity) where the electric field is zero

- 6 Find the potential at a distance 1cm from a proton (B) What is the potential difference between two points that are 1cm and 2cm from a proton? (c) What if ? Repeat part (a) and (b) for an electron
- 7 At a certain distance from point charge, the magnitude of the electric field is 500v/m and the electric potential is -3.00kv
- a) What is the distance to the charge?
- b) What is the magnitude of the charge?8)
- 8) A proton accelerates from rest in a uniform electric field 640N/C its speed is 1.2×10^6 m/s
- a) Find the acceleration of the proton
- b) How long does it take the proton to reach the speed?
- c) How far has it moved in this time?

Short reference material

9.1 Electric Current, Resistance, Ohm's Law and Power

In this section, we study the basic behavior of a flow of electric charges through a region of space. The flow of charges between two points in space is driven by a potential difference between the points. Whenever there is a net flow of charge through some region, an electric current is said to exist. The amount of current depends both on the potential difference and any material that may fill the space through which the charges flow.

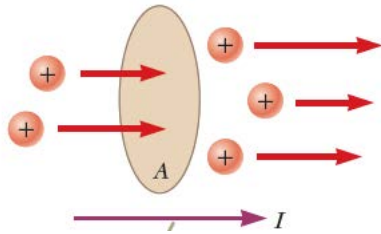


Fig.1 Charges in motion through an area A . The time rate at which charge flows through the area is defined as the current I .

- The **electric current** is defined as:

$$I = \frac{dQ}{dt}$$

- The average current in a conductor is

$$I_{\text{avg}} = nqv_d A$$

where n is the number density of the charge carriers, q is the charge each carrier has, v_d is the **drift speed**, and A is the cross-sectional area.

- The **current density** J through the cross sectional area of the wire is

$$\vec{J} = nq\vec{v}_d$$

- Microscopic **Ohm's law**: the current density is proportional to the electric field, and the constant of proportionality is called **conductivity** σ :

$$\vec{J} = \sigma \vec{E}$$

- The reciprocal of conductivity σ is called **resistivity** ρ :

$$\rho = \frac{1}{\sigma}$$

- Macroscopic Ohm's law: The **resistance** R of a conductor is the ratio of the potential difference ΔV between the two ends of the conductor and the current I :

$$R = \frac{\Delta V}{I}$$

- Resistance is related to resistivity by

$$R = \frac{\rho l}{A}$$

where l is the length and A is the cross-sectional area of the conductor.

- The **drift velocity** of an electron in the conductor is

$$\vec{v}_d = -\frac{e\vec{E}}{m_e}\tau$$

where m_e is the mass of an electron, and τ is the average time between successive collisions.

- The resistivity of a metal is related to τ by

$$\rho = \frac{1}{\sigma} = \frac{m_e}{ne^2\tau}$$

- The temperature variation of resistivity of a conductor is

$$\rho = \rho_0 [1 + \alpha(T - T_0)]$$

where α is the **temperature coefficient of resistivity**.

- Power**, or rate at which energy is delivered to the resistor is

$$P = I\Delta V = I^2 R = \frac{(\Delta V)^2}{R}$$

9.1.1 Solved Problems

1) Resistivity of a Cable: - A 3000-km long cable consists of seven copper wires, each of diameter 0.73 mm, bundled together and surrounded by an insulating sheath. Calculate the resistance of the cable. Use $3 \times 10^{-6} \Omega \cdot \text{cm}$ for the resistivity of the copper.

Solution:

The resistance R of a conductor is related to the resistivity ρ by $R = \rho l / A$, where l and A are the length of the conductor and the cross-sectional area, respectively. Since the cable consists of $N = 7$ copper wires, the total cross sectional area is

$$A = N\pi r^2 = N \frac{\pi d^2}{4} = 7 \frac{\pi (0.073 \text{ cm})^2}{4}$$

The resistance then becomes

$$R = \frac{\rho l}{A} = \frac{(3 \times 10^{-6} \Omega \cdot \text{cm})(3 \times 10^8 \text{ cm})}{7\pi (0.073 \text{ cm})^2 / 4} = 3.1 \times 10^4 \Omega$$

6.5.2 Charge at a Junction

Show that the total amount of charge at the junction of the two materials in Figure 6.5.1 is $\epsilon_0 I (\sigma_2^{-1} - \sigma_1^{-1})$, where I is the current flowing through the junction, and σ_1 and σ_2 are the conductivities for the two materials.

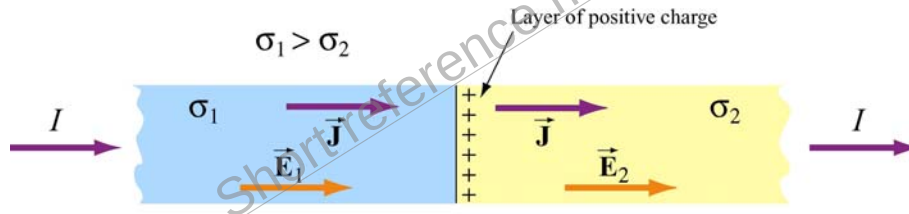


Figure 6.5.1 Charge at a junction.

Solution:

In a steady state of current flow, the normal component of the current density $\vec{J}$ must be the same on both sides of the junction. Since $J = \sigma E$, we have $\sigma_1 E_1 = \sigma_2 E_2$ or

$$E_2 = \left(\frac{\sigma_1}{\sigma_2} \right) E_1$$

Let the charge on the interface be q_{in} , we have, from the Gauss's law:

$$\oiint_S \vec{E} \cdot d\vec{A} = (E_2 - E_1) A = \frac{q_{\text{in}}}{\epsilon_0}$$

or

$$E_2 - E_1 = \frac{q_{\text{in}}}{A\epsilon_0}$$

Substituting the expression for E_2 from above then yields

$$q_{\text{in}} = \epsilon_0 A E_1 \left(\frac{\sigma_1}{\sigma_2} - 1 \right) = \epsilon_0 A \sigma_1 E_1 \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right)$$

Since the current is $I = JA = (\sigma_1 E_1) A$, the amount of charge on the interface becomes

$$q_{\text{in}} = \epsilon_0 I \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right)$$

6.5.3 Drift Velocity

The resistivity of seawater is about $25 \, \Omega \cdot \text{cm}$. The charge carriers are chiefly Na^+ and Cl^- ions, and of each there are about $3 \times 10^{20} / \text{cm}^3$. If we fill a plastic tube 2 meters long with seawater and connect a 12-volt battery to the electrodes at each end, what is the resulting average drift velocity of the ions, in cm/s?

Solution:

The current in a conductor of cross sectional area A is related to the drift speed v_d of the charge carriers by

$$I = enAv_d$$

where n is the number of charges per unit volume. We can then rewrite the Ohm's law as

$$V = IR = (enAv_d) \left(\frac{\rho l}{A} \right) = nev_d \rho l$$

which yields

$$v_d = \frac{V}{ne\rho l}$$

Substituting the values, we have

$$v_d = \frac{12\text{V}}{(6 \times 10^{20} / \text{cm}^3)(1.6 \times 10^{-19} \text{C})(25 \Omega \cdot \text{cm})(200 \text{cm})} = 2.5 \times 10^{-5} \frac{\text{V} \cdot \text{cm}}{\text{C} \cdot \Omega} = 2.5 \times 10^{-5} \frac{\text{cm}}{\text{s}}$$

In converting the units we have used

$$\frac{V}{\Omega \cdot C} = \left(\frac{V}{\Omega} \right) \frac{1}{C} = \frac{A}{C} = s^{-1}$$

6.5.4 Resistance of a Truncated Cone

Consider a material of resistivity ρ in a shape of a truncated cone of altitude h , and radii a and b , for the right and the left ends, respectively, as shown in the Figure 6.5.2.

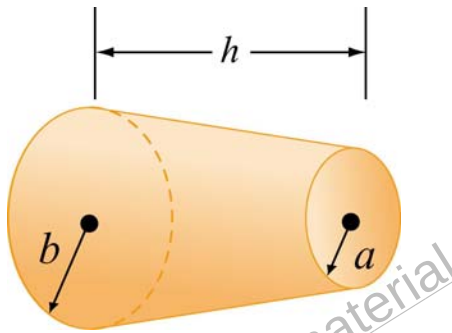


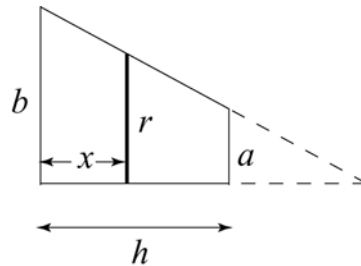
Figure 6.5.2 A truncated Cone.

Assuming that the current is distributed uniformly throughout the cross-section of the cone, what is the resistance between the two ends?

Solution:

Consider a thin disk of radius r at a distance x from the left end. From the figure shown on the right, we have

$$\frac{b-r}{x} = \frac{b-a}{h}$$



or

$$r = (a-b) \frac{x}{h} + b$$

Since resistance R is related to resistivity ρ by $R = \rho l / A$, where l is the length of the conductor and A is the cross section, the contribution to the resistance from the disk having a thickness dy is

$$dR = \frac{\rho dx}{\pi r^2} = \frac{\rho dx}{\pi [b + (a-b)x/h]^2}$$

Straightforward integration then yields

$$R = \int_0^h \frac{\rho dx}{\pi [b + (a-b)x/h]^2} = \frac{\rho h}{\pi ab}$$

where we have used

$$\int \frac{du}{(\alpha u + \beta)^2} = -\frac{1}{\alpha(\alpha u + \beta)}$$

Note that if $b = a$, Eq. (6.2.9) is reproduced.

6.5.5 Resistance of a Hollow Cylinder

Consider a hollow cylinder of length L and inner radius a and outer radius b , as shown in Figure 6.5.3. The material has resistivity ρ .

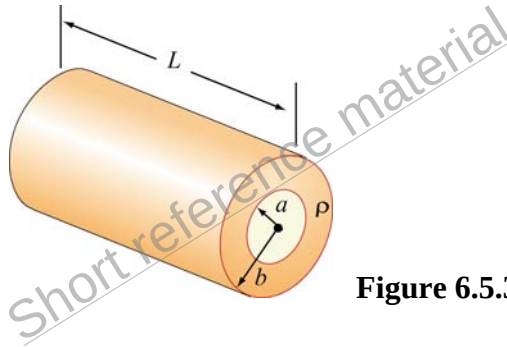


Figure 6.5.3 A hollow cylinder.

- (a) Suppose a potential difference is applied between the ends of the cylinder and produces a current flowing parallel to the axis. What is the resistance measured?
- (b) If instead the potential difference is applied between the inner and outer surfaces so that current flows radially outward, what is the resistance measured?

Solution:

- (a) When a potential difference is applied between the ends of the cylinder, current flows parallel to the axis. In this case, the cross-sectional area is $A = \pi(b^2 - a^2)$, and the resistance is given by

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi(b^2 - a^2)}$$

(b) Consider a differential element which is made up of a thin cylinder of inner radius r and outer radius $r + dr$ and length L . Its contribution to the resistance of the system is given by

$$dR = \frac{\rho dl}{A} = \frac{\rho dr}{2\pi rL}$$

where $A = 2\pi rL$ is the area normal to the direction of current flow. The total resistance of the system becomes

$$R = \int_a^b \frac{\rho dr}{2\pi rL} = \frac{\rho}{2\pi L} \ln\left(\frac{b}{a}\right)$$

6.6 Conceptual Questions

- Two wires A and B of circular cross-section are made of the same metal and have equal lengths, but the resistance of wire A is four times greater than that of wire B. Find the ratio of their cross-sectional areas.
- From the point of view of atomic theory, explain why the resistance of a material increases as its temperature increases.
- Two conductors A and B of the same length and radius are connected across the same potential difference. The resistance of conductor A is twice that of B. To which conductor is more power delivered?

6.7 Additional Problems

6.7.1 Current and Current Density

A sphere of radius 10 mm that carries a charge of $8 \text{ nC} = 8 \times 10^{-9} \text{ C}$ is whirled in a circle at the end of an insulated string. The rotation frequency is $100\pi \text{ rad/s}$.

- What is the basic definition of current in terms of charge?
- What average current does this rotating charge represent?
- What is the average current density over the area traversed by the sphere?

6.7.2 Power Loss and Ohm's Law

A 1500 W radiant heater is constructed to operate at 115 V.

- (a) What will be the current in the heater? [Ans. ~ 10 A]
- (b) What is the resistance of the heating coil? [Ans. ~ 10 Ω]
- (c) How many kilocalories are generated in one hour by the heater? (1 Calorie = 4.18 J)

6.7.3 Resistance of a Cone

A copper resistor of resistivity ρ is in the shape of a cylinder of radius b and length L_1 appended to a truncated right circular cone of length L_2 and end radii b and a as shown in Figure 6.7.1.

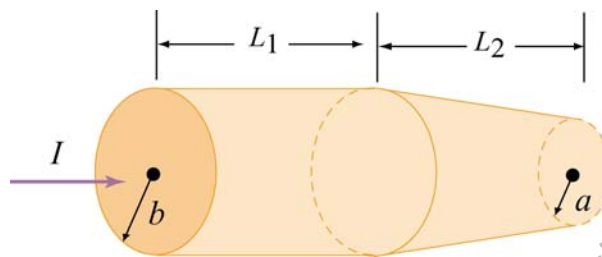


Figure 6.7.1

- (a) What is the resistance of the cylindrical portion of the resistor?
- (b) What is the resistance of the entire resistor? (Hint: For the tapered portion, it is necessary to write down the incremental resistance dR of a small slice, dx , of the resistor at an arbitrary position, x , and then to sum the slices by integration. If the taper is small, one may assume that the current density is uniform across any cross section.)
- (c) Show that your answer reduces to the expected expression if $a = b$.
- (d) If $L_1 = 100$ mm, $L_2 = 50$ mm, $a = 0.5$ mm, $b = 1.0$ mm, what is the resistance?

6.7.4 Current Density and Drift Speed

- (a) A group of charges, each with charge q , moves with velocity $\vec{v}$. The number of particles per unit volume is n . What is the current density $\vec{J}$ of these charges, in magnitude and direction? Make sure that your answer has units of A/m².
- (b) We want to calculate how long it takes an electron to get from a car battery to the starter motor after the ignition switch is turned. Assume that the current flowing is 115 A, and that the electrons travel through copper wire with cross-sectional area 31.2 mm² and length 85.5 cm. What is the current density in the wire? The number density of the conduction electrons in copper is 8.49×10^{28} /m³. Given this number density and the current density, what is the drift speed of the electrons? How long does it take for an

electron starting at the battery to reach the starter motor? [Ans: $3.69 \times 10^6 \text{ A/m}^2$, $2.71 \times 10^{-4} \text{ m/s}$, 52.5 min .]

6.7.5 Current Sheet

A *current sheet*, as the name implies, is a plane containing currents flowing in one direction in that plane. One way to construct a sheet of current is by running many parallel wires in a plane, say the yz -plane, as shown in Figure 6.7.2(a). Each of these wires carries current I out of the page, in the $-\hat{j}$ direction, with n wires per unit length in the z -direction, as shown in Figure 6.7.2(b). Then the current per unit length in the z direction is nI . We will use the symbol K to signify current per unit length, so that $K = nI$ here.

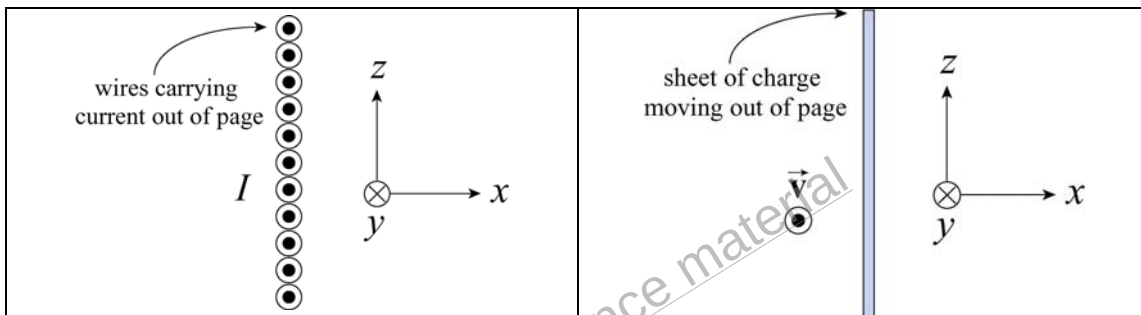


Figure 6.7.2 A current sheet.

Another way to construct a current sheet is to take a non-conducting sheet of charge with fixed charge per unit area σ and move it with some speed in the direction you want current to flow. For example, in the sketch to the left, we have a sheet of charge moving out of the page with speed v . The direction of current flow is out of the page.

(a) Show that the magnitude of the current per unit length in the z direction, K , is given by σv . Check that this quantity has the proper dimensions of current per length. This is in fact a vector relation, $\vec{K}(t) = \sigma \vec{v}(t)$, since the sense of the current flow is in the same direction as the velocity of the positive charges.

(b) A belt transferring charge to the high-potential inner shell of a Van de Graaff accelerator at the rate of 2.83 mC/s . If the width of the belt carrying the charge is 50 cm and the belt travels at a speed of 30 m/s , what is the surface charge density on the belt? [Ans: $189 \mu\text{C/m}^2$]

6.7.6 Resistance and Resistivity

A wire with a resistance of 6.0Ω is drawn out through a die so that its new length is three times its original length. Find the resistance of the longer wire, assuming that the

resistivity and density of the material are not changed during the drawing process. [Ans: 54 Ω].

6.7.7 Power, Current, and Voltage

A 100-W light bulb is plugged into a standard 120-V outlet. (a) How much does it cost per month (31 days) to leave the light turned on? Assume electricity costs 6 cents per $\text{kW} \cdot \text{h}$. (b) What is the resistance of the bulb? (c) What is the current in the bulb? [Ans: (a) \$4.46; (b) 144 Ω ; (c) 0.833 A].

6.7.8 Charge Accumulation at the Interface

Figure 6.7.3 shows a three-layer sandwich made of two resistive materials with resistivities ρ_1 and ρ_2 . From left to right, we have a layer of material with resistivity ρ_1 of width $d/3$, followed by a layer of material with resistivity ρ_2 , also of width $d/3$, followed by another layer of the first material with resistivity ρ_1 , again of width $d/3$.

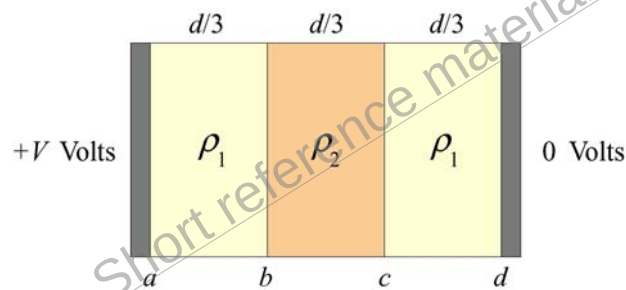


Figure 6.7.3 Charge accumulation at interface.

The cross-sectional area of all of these materials is A . The resistive sandwich is bounded on either side by metallic conductors (black regions). Using a battery (not shown), we maintain a potential difference V across the entire sandwich, between the metallic conductors. The left side of the sandwich is at the higher potential (*i.e.*, the electric fields point from left to right).

There are four interfaces between the various materials and the conductors, which we label a through d , as indicated on the sketch. A steady current I flows through this sandwich from left to right, corresponding to a current density $J = I/A$.

(a) What are the electric fields $\vec{E}_1$ and $\vec{E}_2$ in the two different dielectric materials? To obtain these fields, assume that the current density is the same in every layer. Why must this be true? [Ans: All fields point to the right, $E_1 = \rho_1 I / A$, $E_2 = \rho_2 I / A$; the current densities must be the same in a steady state, otherwise there would be a continuous buildup of charge at the interfaces to unlimited values.]

(b) What is the total resistance R of this sandwich? Show that your expression reduces to the expected result if $\rho_1 = \rho_2 = \rho$. [Ans: $R = d(2\rho_1 + \rho_2)/3A$; if $\rho_1 = \rho_2 = \rho$, then $R = d\rho/A$, as expected.]

(c) As we move from right to left, what are the changes in potential across the three layers, in terms of V and the resistivities? [Ans: $V\rho_1/(2\rho_1 + \rho_2)$, $V\rho_2/(2\rho_1 + \rho_2)$, $V\rho_1/(2\rho_1 + \rho_2)$, summing to a total potential drop of V , as required].

(d) What are the charges per unit area, σ_a through σ_d , at the interfaces? Use Gauss's Law and assume that the electric field in the conducting caps is zero. [Ans: $\sigma_a = -\sigma_d = 3\varepsilon_0 V \rho_1 / d(2\rho_1 + \rho_2)$, $\sigma_b = -\sigma_c = 3\varepsilon_0 V (\rho_2 - \rho_1) / d(2\rho_1 + \rho_2)$.]

(e) Consider the limit $\rho_2 \gg \rho_1$. What do your answers above reduce to in this limit?

Short reference material

7.7 Summary

- The equivalent resistance of a set of resistors connected in series:

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots = \sum_{i=1}^N R_i$$

- The equivalent resistance of a set of resistors connected in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{i=1}^N \frac{1}{R_i}$$

- Kirchhoff's rules:**

(1) The sum of the currents flowing into a junction is equal to the sum of the currents flowing out of the junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}}$$

(2) The algebraic sum of the changes in electric potential in a closed-circuit loop is zero.

$$\sum_{\text{closed loop}} \Delta V = 0$$

- In a charging capacitor, the charges and the current as a function of time are

$$q(t) = Q \left(1 - e^{-\frac{t}{RC}} \right), \quad I(t) = \left(\frac{\mathcal{E}}{R} \right) e^{-t/RC}$$

- In a discharging capacitor, the charges and the current as a function of time are

$$q(t) = Q e^{-t/RC}, \quad I(t) = \left(\frac{Q}{RC} \right) e^{-t/RC}$$

7.8 Problem-Solving Strategy: Applying Kirchhoff's Rules

In this chapter we have seen how Kirchhoff's rules can be used to analyze multiloop circuits. The steps are summarized below:

- (1) Draw a circuit diagram, and label all the quantities, both known and unknown. The number of unknown quantities is equal to the number of linearly independent equations we must look for.
- (2) Assign a direction to the current in each branch of the circuit. (If the actual direction is opposite to what you have assumed, your result at the end will be a negative number.)
- (3) Apply the junction rule to all but one of the junctions. (Applying the junction rule to the last junction will not yield any independent relationship among the currents.)
- (4) Apply the loop rule to the loops until the number of independent equations obtained is the same as the number of unknowns. For example, if there are three unknowns, then we must write down three linearly independent equations in order to have a unique solution.

Traverse the loops using the convention below for ΔV :

resistor	travel direction higher V a $\xrightarrow{I}$ lower V b $\Delta V = V_b - V_a = -IR$	travel direction lower V a $\xleftarrow{I}$ higher V b $\Delta V = V_b - V_a = +IR$
emf source	travel direction lower V a $\xrightarrow{\epsilon}$ higher V b $\Delta V = V_b - V_a = +\epsilon$	travel direction higher V a $\xrightarrow{\epsilon}$ lower V b $\Delta V = V_b - V_a = -\epsilon$
capacitor	travel direction lower V a $\xrightarrow{-q \quad +q}$ higher V b $\Delta V = V_b - V_a = +q/C$	travel direction higher V a $\xrightarrow{+q \quad -q}$ lower V b $\Delta V = V_b - V_a = -q/C$

The same equation is obtained whether the closed loop is traversed clockwise or counterclockwise. (The expressions actually differ by an overall negative sign. However, using the loop rule, we are led to $0 = -0$, and hence the same equation.)

- (5) Solve the simultaneous equations to obtain the solutions for the unknowns.

As an example of illustrating how the above procedures are executed, let's analyze the circuit shown in Figure 7.8.1.

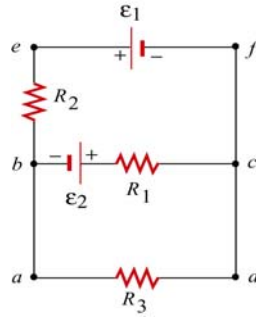


Figure 7.8.1 A multiloop circuit.

Suppose the emf sources ε_1 and ε_2 , and the resistances R_1 , R_2 and R_3 are all given, and we would like to find the currents through each resistor, using the methodology outlined above.

- (1) The unknown quantities are the three currents I_1 , I_2 and I_3 , associated with the three resistors. Therefore, to solve the system, we must look for three independent equations.
- (2) The directions for the three currents are arbitrarily assigned, as indicated in Figure 7.8.2.

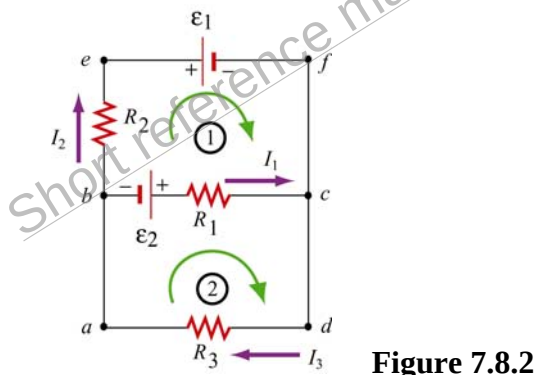


Figure 7.8.2

- (3) Applying Kirchhoff's current rule to junction b yields

$$I_1 + I_2 = I_3$$

since I_1 and I_2 are leaving the junction while I_3 is entering the junction. The same equation is obtained if we consider junction c .

- (4) The other two equations can be obtained by using the loop (voltage) rule, which states that the net potential difference across all elements in a closed circuit loop is zero. Traversing the first loop $befcb$ in the clockwise direction yields

$$-I_2 R_2 - \varepsilon_1 + I_1 R_1 - \varepsilon_2 = 0$$

Similarly, traversing the second loop $abcda$ clockwise gives

$$\varepsilon_2 - I_1 R_1 - I_3 R_3 = 0$$

Note however, that one may also consider the big loop $abefcda$. This leads to

$$-I_2 R_2 - \varepsilon_1 - I_3 R_3 = 0$$

However, the equation is not linearly independent of the other two loop equations since it is simply the sum of those equations.

(5) The solutions to the above three equations are given by, after tedious but straightforward algebra,

$$I_1 = \frac{\varepsilon_1 R_3 + \varepsilon_2 R_3 + \varepsilon_2 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_2 = -\frac{\varepsilon_1 R_1 + \varepsilon_1 R_3 + \varepsilon_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$I_3 = \frac{\varepsilon_2 R_2 - \varepsilon_1 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

Note that I_2 is a negative quantity. This simply indicates that the direction of I_2 is opposite of what we have initially assumed.

7.9 Solved Problems

7.9.1 Equivalent Resistance

Consider the circuit shown in Figure 7.9.1. For a given resistance R_0 , what must be the value of R_1 so that the equivalent resistance between the terminals is equal to R_0 ?

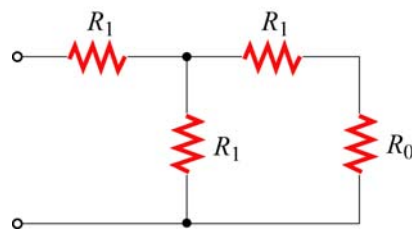


Figure 7.9.1

Solution:

The equivalent resistance, R' , due to the three resistors on the right is

$$\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_0 + R_1} = \frac{R_0 + 2R_1}{R_1(R_0 + R_1)}$$

or

$$R' = \frac{R_1(R_0 + R_1)}{R_0 + 2R_1}$$

Since R' is in series with the fourth resistor R_1 , the equivalent resistance of the entire configuration becomes

$$R_{eq} = R_1 + \frac{R_1(R_0 + R_1)}{R_0 + 2R_1} = \frac{3R_1^2 + 2R_1R_0}{R_0 + 2R_1}$$

If $R_{eq} = R_0$, then

$$R_0(R_0 + 2R_1) = 3R_1^2 + 2R_1R_0 \Rightarrow R_0^2 = 3R_1^2$$

or

$$R_1 = \frac{R_0}{\sqrt{3}}$$

7.9.2 Variable Resistance

Show that, if a battery of fixed emf $\mathcal{E}$ and internal resistance r is connected to a variable external resistance R , the maximum power is delivered to the external resistor when $R = r$.

Solution:

Using Kirchhoff's rule,

$$\mathcal{E} = I(R + r)$$

which implies

$$I = \frac{\mathcal{E}}{R + r}$$

The power dissipated is equal to

$$P = I^2 R = \frac{\mathcal{E}^2}{(R + r)^2} R$$

To find the value of R which gives out the maximum power, we differentiate P with respect to R and set the derivative equal to 0:

$$\frac{dP}{dR} = \epsilon^2 \left[\frac{1}{(R+r)^2} - \frac{2R}{(R+r)^3} \right] = \epsilon^2 \frac{r-R}{(R+r)^3} = 0$$

which implies

$$R = r$$

This is an example of “impedance matching,” in which the variable resistance R is adjusted so that the power delivered to it is maximized. The behavior of P as a function of R is depicted in Figure 7.9.2 below.

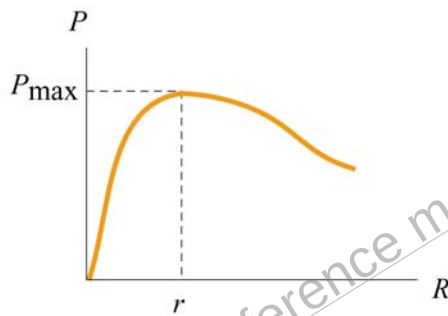


Figure 7.9.2

7.9.3 RC Circuit

In the circuit in figure 7.9.3, suppose the switch has been open for a very long time. At time $t = 0$, it is suddenly closed.

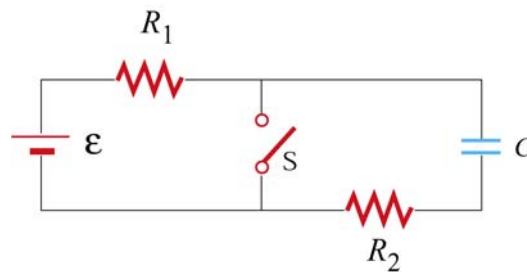


Figure 7.9.3

- What is the time constant before the switch is closed?
- What is the time constant after the switch is closed?
- Find the current through the switch as a function of time after the switch is closed.

Solutions:

(a) Before the switch is closed, the two resistors R_1 and R_2 are in series with the capacitor. Since the equivalent resistance is $R_{eq} = R_1 + R_2$, the time constant is given by

$$\tau = R_{eq}C = (R_1 + R_2)C$$

The amount of charge stored in the capacitor is

$$q(t) = C\varepsilon(1 - e^{-t/\tau})$$

(b) After the switch is closed, the closed loop on the right becomes a decaying RC circuit with time constant $\tau' = R_2C$. Charge begins to decay according to

$$q'(t) = C\varepsilon e^{-t/\tau'}$$

(c) The current passing through the switch consists of two sources: the steady current I_1 from the left circuit, and the decaying current I_2 from the RC circuit. The currents are given by

$$I_1 = \frac{\varepsilon}{R_1}$$

$$I'(t) = \frac{dq'}{dt} = -\left(\frac{C\varepsilon}{\tau'}\right)e^{-t/\tau'} = -\left(\frac{\varepsilon}{R_2}\right)e^{-t/R_2C}$$

The negative sign in $I'(t)$ indicates that the direction of flow is opposite of the charging process. Thus, since both I_1 and I' move downward across the switch, the total current is

$$I(t) = I_1 + I'(t) = \frac{\varepsilon}{R_1} + \left(\frac{\varepsilon}{R_2}\right)e^{-t/R_2C}$$

7.9.4 Parallel vs. Series Connections

Figure 7.9.4 show two resistors with resistances R_1 and R_2 connected in parallel and in series. The battery has a terminal voltage of ε .

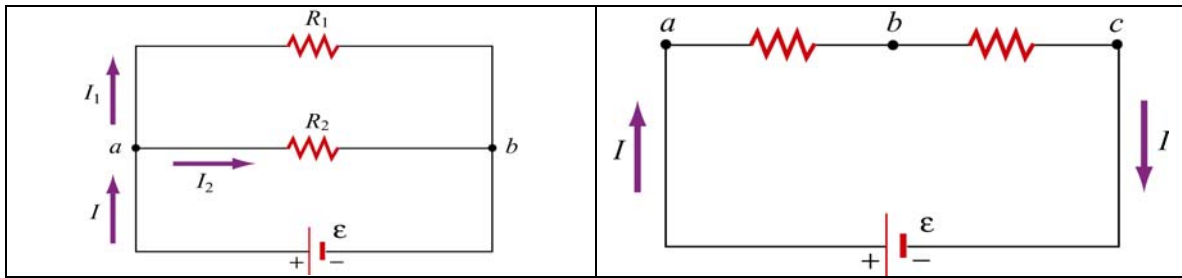


Figure 7.9.4

Suppose R_1 and R_2 are connected in parallel.

- (a) Find the power delivered to each resistor.
- (b) Show that the sum of the power used by each resistor is equal to the power supplied by the battery.

Suppose R_1 and R_2 are now connected in series.

- (c) Find the power delivered to each resistor.
- (d) Show that the sum of the power used by each resistor is equal to the power supplied by the battery.
- (e) Which configuration, parallel or series, uses more power?

Solutions:

- (a) When two resistors are connected in parallel, the current through each resistor is

$$I_1 = \frac{\mathcal{E}}{R_1}, \quad I_2 = \frac{\mathcal{E}}{R_2}$$

and the power delivered to each resistor is given by

$$P_1 = I_1^2 R_1 = \frac{\mathcal{E}^2}{R_1}, \quad P_2 = I_2^2 R_2 = \frac{\mathcal{E}^2}{R_2}$$

The results indicate that the smaller the resistance, the greater the amount of power delivered. If the loads are the light bulbs, then the one with smaller resistance will be brighter since more power is delivered to it.

- (b) The total power delivered to the two resistors is

$$P_R = P_1 + P_2 = \frac{\mathcal{E}^2}{R_1} + \frac{\mathcal{E}^2}{R_2} = \frac{\mathcal{E}^2}{R_{\text{eq}}}$$

where

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}$$

is the equivalent resistance of the circuit. On the other hand, the total power supplied by the battery is $P_{\mathcal{E}} = I\mathcal{E}$, where $I = I_1 + I_2$, as seen from the figure. Thus,

$$P_{\mathcal{E}} = I_1 \mathcal{E} + I_2 \mathcal{E} = \left(\frac{\mathcal{E}}{R_1} \right) \mathcal{E} + \left(\frac{\mathcal{E}}{R_2} \right) \mathcal{E} = \frac{\mathcal{E}^2}{R_1} + \frac{\mathcal{E}^2}{R_2} = \frac{\mathcal{E}^2}{R_{\text{eq}}} = P_R$$

as required by energy conservation.

(c) When the two resistors are connected in series, the equivalent resistance becomes

$$R'_{\text{eq}} = R_1 + R_2$$

and the currents through the resistors are

$$I_1 = I_2 = I = \frac{\mathcal{E}}{R_1 + R_2}$$

Therefore, the power delivered to each resistor is

$$P_1 = I_1^2 R_1 = \left(\frac{\mathcal{E}}{R_1 + R_2} \right)^2 R_1, \quad P_2 = I_2^2 R_2 = \left(\frac{\mathcal{E}}{R_1 + R_2} \right)^2 R_2$$

Contrary to what we have seen in the parallel case, when connected in series, the greater the resistance, the greater the fraction of the power delivered. Once again, if the loads are light bulbs, the one with greater resistance will be brighter.

(d) The total power delivered to the resistors is

$$P'_R = P_1 + P_2 = \left(\frac{\mathcal{E}}{R_1 + R_2} \right)^2 R_1 + \left(\frac{\mathcal{E}}{R_1 + R_2} \right)^2 R_2 = \frac{\mathcal{E}^2}{R_1 + R_2} = \frac{\mathcal{E}^2}{R'_{\text{eq}}}$$

On the other hand, the power supplied by the battery is

$$P'_\varepsilon = I\varepsilon = \left(\frac{\varepsilon}{R_1 + R_2} \right) \varepsilon = \frac{\varepsilon^2}{R_1 + R_2} = \frac{\varepsilon^2}{R'_{\text{eq}}}$$

Again, we see that $P'_\varepsilon = P_R$, as required by energy conservation.

(e) Comparing the results obtained in (b) and (d), we see that

$$P_\varepsilon = \frac{\varepsilon^2}{R_1} + \frac{\varepsilon^2}{R_2} > \frac{\varepsilon^2}{R_1 + R_2} = P'_\varepsilon$$

which means that the parallel connection uses more power. The equivalent resistance of two resistors connected in parallel is always smaller than that connected in series.

7.9.5 Resistor Network

Consider a cube which has identical resistors with resistance R along each edge, as shown in Figure 7.9.5.

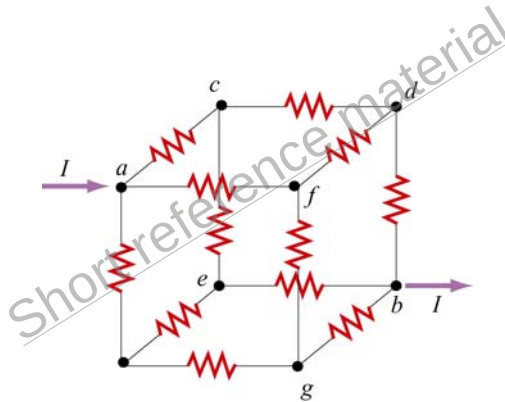


Figure 7.9.5 Resistor network

Show that the equivalent resistance between points a and b is $R_{\text{eq}} = 5R/6$.

Solution:

From symmetry arguments, the current which enters a must split evenly, with $I/3$ going to each branch. At the next junction, say c , $I/3$ must further split evenly with $I/6$ going through the two paths ce and cd . The current going through the resistor in db is the sum of the currents from fd and cd : $I/6 + I/6 = I/3$.

Thus, the potential difference between a and b can be obtained as

$$V_{ab} = V_{ac} + V_{cd} + V_{db} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}IR$$

which shows that the equivalent resistance is

$$R_{\text{eq}} = \frac{5}{6}R$$

7.10 Conceptual Questions

1. Given three resistors of resistances R_1 , R_2 and R_3 , how should they be connected to (a) maximize (b) minimize the equivalent resistance?
2. Why do the headlights on the car become dim when the car is starting?
3. Does the resistor in an RC circuit affect the maximum amount of charge that can be stored in a capacitor? Explain.
4. Can one construct a circuit such that the potential difference across the terminals of the battery is zero? Explain.

7.11 Additional Problems

7.11.1 Resistive Circuits

Consider two identical batteries of emf $\mathcal{E}$ and internal resistance r . They may be connected in series or in parallel and are used to establish a current in resistance R as shown in Figure 7.11.1.

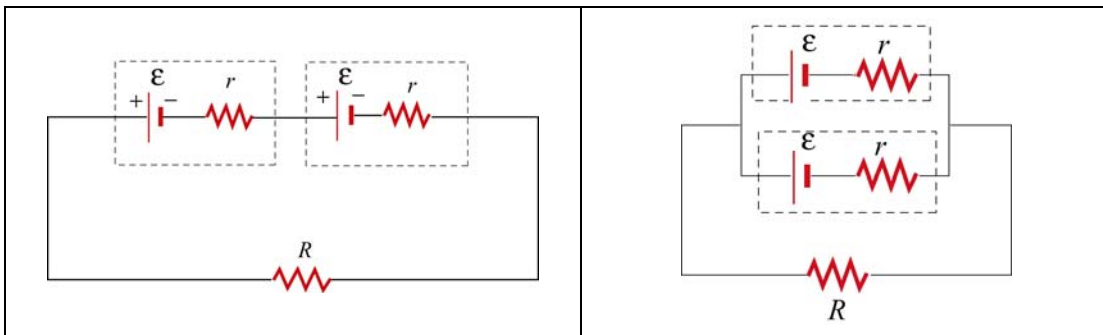


Figure 7.11.1 Two batteries connected in (a) series, and (b) parallel.

(a) Derive an expression for the current in R for the series connection shown in Figure 7.11.1(a). Be sure to indicate the current on the sketch (to establish a sign convention for the direction) and apply Kirchhoff's loop rule.

(b) Find the current for the parallel connection shown in Figure 7.11.1(b).

(c) For what relative values of r and R would the currents in the two configurations be the same?; be larger in Figure 7.11.1(a)?; be larger in 7.11.1(b)?

7.11.2 Multiloop Circuit

Consider the circuit shown in Figure 7.11.2. Neglecting the internal resistance of the batteries, calculate the currents through each of the three resistors.

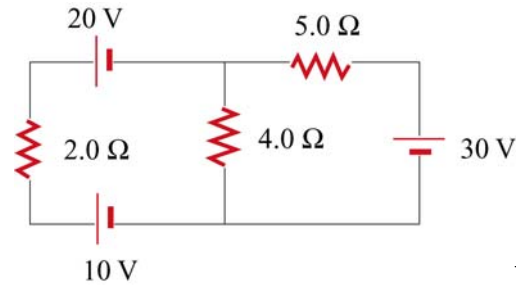


Figure 7.11.2

7.11.3 Power Delivered to the Resistors

Consider the circuit shown in Figure 7.11.3. Find the power delivered to each resistor.

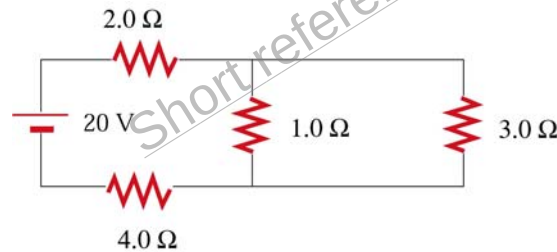


Figure 7.11.3

7.11.4 Resistor Network

Consider an infinite network of resistors of resistances R_0 and R_1 shown in Figure 7.11.4. Show that the equivalent resistance of this network is

$$R_{\text{eq}} = R_1 + \sqrt{R_1^2 + 2R_1R_0}$$

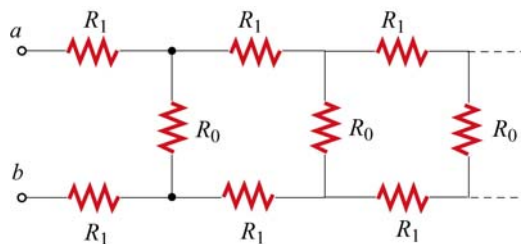


Figure 7.11.4

7.11.5 RC Circuit

Consider the circuit shown in Figure 7.11.5. Let $\mathcal{E} = 40 \text{ V}$, $R_1 = 8.0 \Omega$, $R_2 = 6.0 \Omega$, $R_3 = 4.0 \Omega$ and $C = 4.0 \mu\text{F}$. The capacitor is initially uncharged.

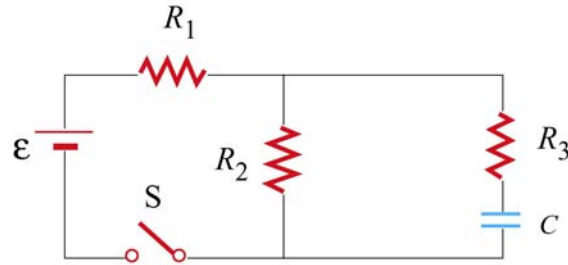


Figure 7.11.5

At $t = 0$, the switch is closed.

- Find the current through each resistor immediately after the switch is closed.
- Find the final charge on the capacitor.

7.11.6 Resistors in Series and Parallel

A circuit containing five resistors and a 12 V battery is shown in Figure 7.11.6. Find the potential drop across the 5Ω resistor. [Ans: 7.5 V].

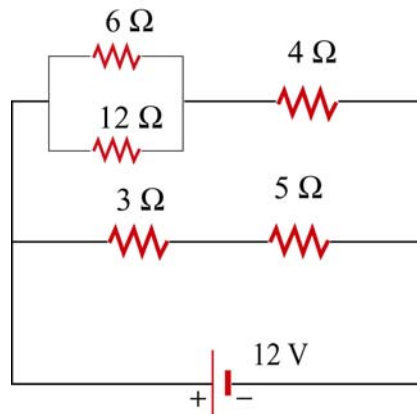


Figure 7.11.6

8.7 Summary

- The **magnetic force** acting on a charge q traveling at a velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

- The magnetic force acting on a wire of length $\vec{\ell}$ carrying a steady current I in a magnetic field $\vec{B}$ is

$$\vec{F}_B = I\vec{\ell} \times \vec{B}$$

- The magnetic force $d\vec{F}_B$ generated by a small portion of current I of length $d\vec{s}$ in a magnetic field $\vec{B}$ is

$$d\vec{F}_B = I d\vec{s} \times \vec{B}$$

- The **torque** $\vec{\tau}$ acting on a close loop of wire of area A carrying a current I in a uniform magnetic field $\vec{B}$ is

$$\vec{\tau} = I\vec{A} \times \vec{B}$$

where $\vec{A}$ is a vector which has a magnitude of A and a direction perpendicular to the loop.

- The **magnetic dipole moment** of a closed loop of wire of area A carrying a current I is given by

$$\vec{\mu} = I\vec{A}$$

- The torque exerted on a magnetic dipole $\vec{\mu}$ placed in an external magnetic field $\vec{B}$ is

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

- The potential energy of a magnetic dipole placed in a magnetic field is

$$U = -\vec{\mu} \cdot \vec{B}$$

- If a particle of charge q and mass m enters a magnetic field of magnitude B with a velocity $\vec{v}$ perpendicular to the magnetic field lines, the radius of the circular path that the particle follows is given by

$$r = \frac{mv}{|q|B}$$

and the angular speed of the particle is

$$\omega = \frac{|q|B}{m}$$

8.8 Problem-Solving Tips

In this Chapter, we have shown that in the presence of both magnetic field $\vec{B}$ and the electric field $\vec{E}$, the total force acting on a moving particle with charge q is $\vec{F} = \vec{F}_e + \vec{F}_B = q(\vec{E} + \vec{v} \times \vec{B})$, where $\vec{v}$ is the velocity of the particle. The direction of $\vec{F}_B$ involves the cross product of $\vec{v}$ and $\vec{B}$, based on the right-hand rule. In Cartesian coordinates, the unit vectors are $\hat{i}$, $\hat{j}$ and $\hat{k}$ which satisfy the following properties:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

For $\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$, the cross product may be obtained as

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = (v_y B_z - v_z B_y) \hat{i} + (v_z B_x - v_x B_z) \hat{j} + (v_x B_y - v_y B_x) \hat{k}$$

If only the magnetic field is present, and $\vec{v}$ is perpendicular to $\vec{B}$, then the trajectory is a circle with a radius $r = mv / |q| B$, and an angular speed $\omega = |q| B / m$.

When dealing with a more complicated case, it is useful to work with individual force components. For example,

$$F_x = ma_x = qE_x + q(v_y B_z - v_z B_y)$$

8.9 Solved Problems

8.9.1 Rolling Rod

A rod with a mass m and a radius R is mounted on two parallel rails of length a separated by a distance ℓ , as shown in the Figure 8.9.1. The rod carries a current I and rolls without slipping along the rails which are placed in a uniform magnetic field $\vec{B}$ directed into the page. If the rod is initially at rest, what is its speed as it leaves the rails?

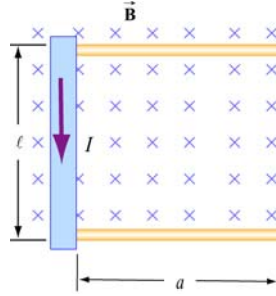
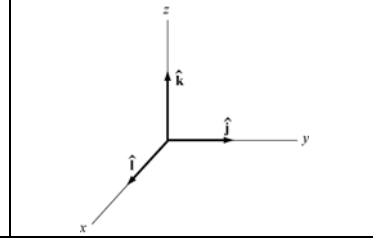


Figure 8.9.1 Rolling rod in uniform magnetic field

Solution:

Using the coordinate system shown on the right, the magnetic force acting on the rod is given by

$$\vec{F}_B = I\vec{\ell} \times \vec{B} = I(\ell\hat{i}) \times (-B\hat{k}) = I\ell B\hat{j} \quad (8.9.1)$$



The total work done by the magnetic force on the rod as it moves through the region is

$$W = \int \vec{F}_B \cdot d\vec{s} = F_B a = (I\ell B)a \quad (8.9.2)$$

By the work-energy theorem, W must be equal to the change in kinetic energy:

$$\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad (8.9.3)$$

where both translation and rolling are involved. Since the moment of inertia of the rod is given by $I = mR^2/2$, and the condition of rolling with slipping implies $\omega = v/R$, we have

$$I\ell Ba = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2 \quad (8.9.4)$$

Thus, the speed of the rod as it leaves the rails is

$$v = \sqrt{\frac{4I\ell Ba}{3m}} \quad (8.9.5)$$

8.9.2 Suspended Conducting Rod

A conducting rod having a mass density λ kg/m is suspended by two flexible wires in a uniform magnetic field $\vec{B}$ which points out of the page, as shown in Figure 8.9.2.

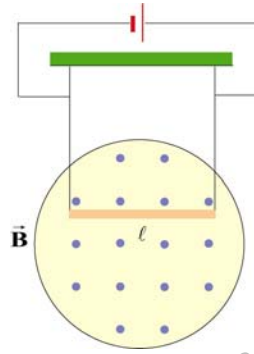
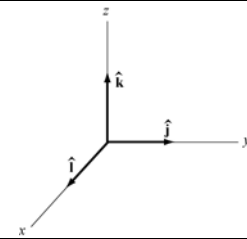


Figure 8.9.2 Suspended conducting rod in uniform magnetic field

If the tension on the wires is zero, what are the magnitude and the direction of the current in the rod?

Solution:

In order that the tension in the wires be zero, the magnetic force $\vec{F}_B = I\vec{\ell} \times \vec{B}$ acting on the conductor must exactly cancel the downward gravitational force $\vec{F}_g = -mg\hat{k}$.



For $\vec{F}_B$ to point in the $+z$ -direction, we must have $\vec{\ell} = -\ell\hat{j}$, i.e., the current flows to the left, so that

$$\vec{F}_B = I\vec{\ell} \times \vec{B} = I(-\ell\hat{j}) \times (B\hat{i}) = -I\ell B(\hat{j} \times \hat{i}) = +I\ell B\hat{k} \quad (8.9.6)$$

The magnitude of the current can be obtain from

$$I\ell B = mg \quad (8.9.7)$$

or

$$I = \frac{mg}{B\ell} = \frac{\lambda g}{B} \quad (8.9.8)$$

8.9.3 Charged Particles in Magnetic Field

Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B , are accelerated from rest by a potential difference ΔV , and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Solution:

The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V \quad (8.9.9)$$

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}} \quad (8.9.10)$$

The charges move in semicircles, since the magnetic force points radially inward and provides the source of the centripetal force:

$$\frac{mv^2}{r} = qvB \quad (8.9.11)$$

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB} \sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B} \sqrt{\frac{2m\Delta V}{q}} \quad (8.9.12)$$

which shows that r is proportional to $(m/q)^{1/2}$. The mass ratio can then be obtained from

$$\frac{r_A}{r_B} = \frac{(m_A/q_A)^{1/2}}{(m_B/q_B)^{1/2}} \Rightarrow \frac{R}{2R} = \frac{(m_A/q)^{1/2}}{(m_B/2q)^{1/2}} \quad (8.9.13)$$

which gives

$$\frac{m_A}{m_B} = \frac{1}{8} \quad (8.9.14)$$

8.9.4 Bar Magnet in Non-Uniform Magnetic Field

A bar magnet with its north pole up is placed along the symmetric axis below a horizontal conducting ring carrying current I , as shown in the Figure 8.9.3. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

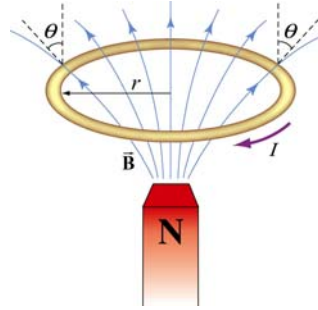


Figure 8.9.3 A bar magnet approaching a conducting ring

Solution:

The magnetic force acting on a small differential current-carrying element $I d\vec{s}$ on the ring is given by $d\vec{F}_B = I d\vec{s} \times \vec{B}$, where $\vec{B}$ is the magnetic field due to the bar magnet. Using cylindrical coordinates $(\hat{r}, \hat{\phi}, \hat{z})$ as shown in Figure 8.9.4, we have

$$d\vec{F}_B = I(-ds \hat{\phi}) \times (B \sin \theta \hat{r} + B \cos \theta \hat{z}) = (IBds) \sin \theta \hat{z} - (IBds) \cos \theta \hat{r} \quad (8.9.15)$$

Due to the axial symmetry, the radial component of the force will exactly cancel, and we are left with the z-component.

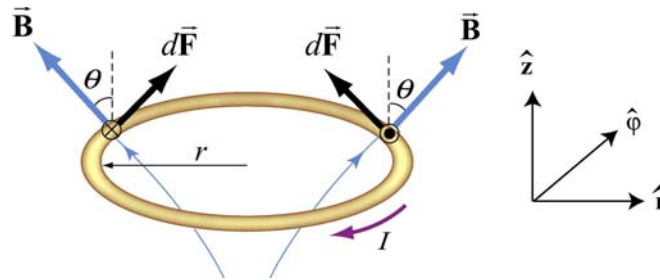


Figure 8.9.4 Magnetic force acting on the conducting ring

The total force acting on the ring then becomes

$$\vec{F}_B = (IB \sin \theta) \hat{z} \oint ds = (2\pi r IB \sin \theta) \hat{z} \quad (8.9.16)$$

The force points in the $+z$ direction and therefore is repulsive.

8.10 Conceptual Questions

1. Can a charged particle move through a uniform magnetic field without experiencing any force? Explain.
2. If no work can be done on a charged particle by the magnetic field, how can the motion of the particle be influenced by the presence of a field?
3. Suppose a charged particle is moving under the influence of both electric and magnetic fields. How can the effect of the two fields on the motion of the particle be distinguished?
4. What type of magnetic field can exert a force on a magnetic dipole? Is the force repulsive or attractive?
5. If a compass needle is placed in a uniform magnetic field, is there a net magnetic force acting on the needle? Is there a net torque?

8.11 Additional Problems

8.11.1 Force Exerted by a Magnetic Field

The electrons in the beam of television tube have an energy of 12 keV ($1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$). The tube is oriented so that the electrons move horizontally from south to north. At MIT, the Earth's magnetic field points roughly vertically down (i.e. neglect the component that is directed toward magnetic north) and has magnitude $B \sim 5 \times 10^{-5} \text{ T}$.

- (a) In what direction will the beam deflect?
- (b) What is the acceleration of a given electron associated with this deflection? [Ans. $\sim 10^{-15} \text{ m/s}^2$.]
- (c) How far will the beam deflect in moving 0.20 m through the television tube?

8.11.2 Magnetic Force on a Current Carrying Wire

A square loop of wire, of length $\ell = 0.1 \text{ m}$ on each side, has a mass of 50 g and pivots about an axis AA' that corresponds to a horizontal side of the square, as shown in Figure 8.11.1. A magnetic field of 500 G, directed vertically downward, uniformly fills the region in the vicinity of the loop. The loop carries a current I so that it is in equilibrium at $\theta = 20^\circ$.

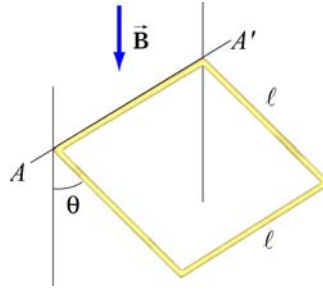


Figure 8.11.1 Magnetic force on a current-carrying square loop.

- Consider the force on each segment separately and find the direction of the current that flows in the loop to maintain the 20° angle.
- Calculate the torque about the axis due to these forces.
- Find the current in the loop by requiring the sum of all torques (about the axis) to be zero. (Hint: Consider the effect of gravity on each of the 4 segments of the wire separately.) [Ans. $I \sim 20$ A.]
- Determine the magnitude and direction of the force exerted on the axis by the pivots.
- Repeat part (b) by now using the definition of a magnetic dipole to calculate the torque exerted on such a loop due to the presence of a magnetic field.

8.11.3 Sliding Bar

A conducting bar of length ℓ is placed on a frictionless inclined plane which is tilted at an angle θ from the horizontal, as shown in Figure 8.11.2.

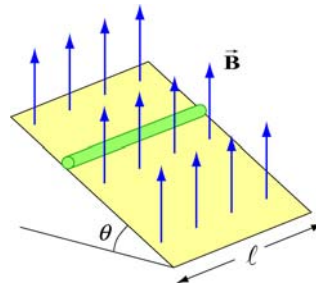


Figure 8.11.2 Magnetic force on a conducting bar

A uniform magnetic field is applied in the vertical direction. To prevent the bar from sliding down, a voltage source is connected to the ends of the bar with current flowing through. Determine the magnitude and the direction of the current such that the bar will remain stationary.

8.11.4 Particle Trajectory

A particle of charge $-q$ is moving with a velocity $\vec{v}$. It then enters midway between two plates where there exists a uniform magnetic field pointing into the page, as shown in Figure 8.11.3.

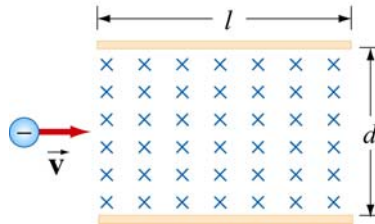


Figure 8.11.3 Charged particle moving under the influence of a magnetic field

- Is the trajectory of the particle deflected upward or downward?
- Compute the distance between the left end of the plate and where the particle strikes.

8.11.5 Particle Orbits in a Magnetic Field

Suppose the entire x - y plane to the right of the origin O is filled with a uniform magnetic field $\vec{B}$ pointing out of the page, as shown in Figure 8.11.4.

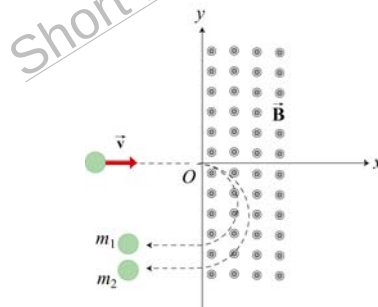


Figure 8.11.4

Two charged particles travel along the negative x axis in the positive x direction, each with speed v , and enter the magnetic field at the origin O . The two particles have the same charge q , but have different masses, m_1 and m_2 . When in the magnetic field, their trajectories both curve in the same direction, but describe semi-circles with different radii. The radius of the semi-circle traced out by particle 2 is exactly *twice* as big as the radius of the semi-circle traced out by particle 1.

- Is the charge q of these particles such that $q > 0$, or is $q < 0$?

(b) *Derive* (do not simply state) an expression for the radius R_1 of the semi-circle traced out by particle 1, in terms of q , v , B , and m_1 .

(c) What is the ratio m_2 / m_1 ?

(d) Is it possible to apply an electric field $\vec{E}$ in the region $x > 0$ only which will cause both particles to continue to move in a straight line after they enter the region $x > 0$? If so, indicate the magnitude and direction of that electric field, in terms of the quantities given. If not, why not?

8.11.6 Force and Torque on a Current Loop

A current loop consists of a semicircle of radius R and two straight segments of length ℓ with an angle θ between them. The loop is then placed in a uniform magnetic field pointing to the right, as shown in Figure 8.11.5.

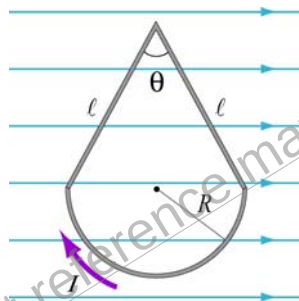


Figure 8.11.5 Current loop placed in a uniform magnetic field

- Find the net force on the current loop.
- Find the net torque on the current loop.

8.11.7 Force on a Wire

A straight wire of length 0.2 m carries a 7.0 A current. It is immersed in a uniform magnetic field of 0.1 T whose direction lies 20 degrees from the direction of the current.

- What is the direction of the force on the wire? Make a sketch to show your answer.
- What is the magnitude of the force? [Ans. ~ 0.05 N]
- How could you maximize the force without changing the field or current?

8.11.8 Levitating Wire

A copper wire of diameter d carries a current density $\vec{J}$ at the Earth's equator where the Earth's magnetic field is horizontal, points north, and has magnitude $B = 0.5 \times 10^{-4} \text{ T}$. The wire lies in a plane that is parallel to the surface of the Earth and is oriented in the east-west direction. The density and resistivity of copper are $\rho_m = 8.9 \times 10^3 \text{ kg/m}^3$ and $\rho = 1.7 \times 10^{-8} \Omega \cdot \text{m}$, respectively.

(a) How large must $\vec{J}$ be, and which direction must it flow in order to levitate the wire? Use $g = 9.8 \text{ m/s}^2$

(b) When the wire is floating how much power will be dissipated per cubic centimeter?

Short reference material

phenomenon is known as *hysteresis*. The variation of $\vec{M}$ as a function of the externally applied field $\vec{B}_0$ is shown in Figure 9.6.4. The loop $abcdef$ is a *hysteresis curve*.

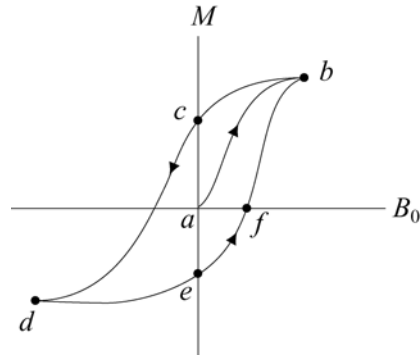


Figure 9.6.4 A hysteresis curve.

Moreover, in ferromagnets, the strong interaction between neighboring atomic dipole moments can keep those dipole moments aligned, even when the external magnet field is reduced to zero. And these aligned dipoles can thus produce a strong magnetic field, all by themselves, without the necessity of an external magnetic field. This is the origin of permanent magnets. To see how strong such magnets can be, consider the fact that magnetic dipole moments of atoms typically have magnitudes of the order of $10^{-23} \text{ A} \cdot \text{m}^2$. Typical atomic densities are $10^{29} \text{ atoms/m}^3$. If all these dipole moments are aligned, then we would get a magnetization of order

$$M \sim (10^{-23} \text{ A} \cdot \text{m}^2)(10^{29} \text{ atoms/m}^3) \sim 10^6 \text{ A/m} \quad (9.6.12)$$

The magnetization corresponds to values of $\vec{B}_M = \mu_0 \vec{M}$ of order 1 tesla, or 10,000 Gauss, just due to the atomic currents alone. This is how we get permanent magnets with fields of order 2200 Gauss.

9.7 Summary

- **Biot-Savart law** states that the magnetic field $d\vec{B}$ at a point due to a length element $d\vec{s}$ carrying a steady current I and located at $\vec{r}$ away is given by

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{s} \times \hat{r}}{r^2}$$

where $r = |\vec{r}|$ and $\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}$ is the permeability of free space.

- The magnitude of the magnetic field at a distance r away from an infinitely long straight wire carrying a current I is

$$B = \frac{\mu_0 I}{2\pi r}$$

- The magnitude of the magnetic force F_B between two straight wires of length ℓ carrying steady current of I_1 and I_2 and separated by a distance r is

$$F_B = \frac{\mu_0 I_1 I_2 \ell}{2\pi r}$$

- Ampere's law** states that the line integral of $\vec{B} \cdot d\vec{s}$ around any closed loop is proportional to the total steady current passing through any surface that is bounded by the close loop:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}}$$

- The magnetic field inside a **toroid** which has N closely spaced of wire carrying a current I is given by

$$B = \frac{\mu_0 NI}{2\pi r}$$

where r is the distance from the center of the toroid.

- The magnetic field inside a **solenoid** which has N closely spaced of wire carrying current I in a length of l is given by

$$B = \mu_0 \frac{N}{l} I = \mu_0 nI$$

where n is the number of number of turns per unit length.

- The properties of magnetic materials are as follows:

Materials	Magnetic susceptibility χ_m	Relative permeability $\kappa_m = 1 + \chi_m$	Magnetic permeability $\mu_m = \kappa_m \mu_0$
Diamagnetic	$-10^{-5} \sim -10^{-9}$	$\kappa_m < 1$	$\mu_m < \mu_0$
Paramagnetic	$10^{-5} \sim 10^{-3}$	$\kappa_m > 1$	$\mu_m > \mu_0$
Ferromagnetic	$\chi_m \gg 1$	$\kappa_m \gg 1$	$\mu_m \gg \mu_0$

9.8 Appendix 1: Magnetic Field off the Symmetry Axis of a Current Loop

In Example 9.2 we calculated the magnetic field due to a circular loop of radius R lying in the xy plane and carrying a steady current I , at a point P along the axis of symmetry. Let's see how the same technique can be extended to calculating the field at a point off the axis of symmetry in the yz plane.

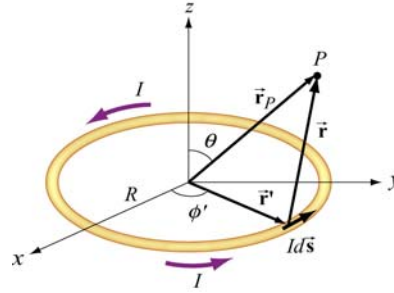


Figure 9.8.1 Calculating the magnetic field off the symmetry axis of a current loop.

Again, as shown in Example 9.1, the differential current element is

$$Id\vec{s} = R d\phi'(-\sin\phi'\hat{i} + \cos\phi'\hat{j})$$

and its position is described by $\vec{r}' = R(\cos\phi'\hat{i} + \sin\phi'\hat{j})$. On the other hand, the field point P now lies in the yz plane with $\vec{r}_p = y\hat{j} + z\hat{k}$, as shown in Figure 9.8.1. The corresponding relative position vector is

$$\vec{r} = \vec{r}_p - \vec{r}' = -R\cos\phi'\hat{i} + (y - R\sin\phi')\hat{j} + z\hat{k} \quad (9.8.1)$$

with a magnitude

$$r = |\vec{r}| = \sqrt{(-R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2} = \sqrt{R^2 + y^2 + z^2 - 2yR\sin\phi} \quad (9.8.2)$$

and the unit vector

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r}_p - \vec{r}'}{|\vec{r}_p - \vec{r}'|}$$

pointing from $Id\vec{s}$ to P . The cross product $d\vec{s} \times \hat{r}$ can be simplified as

$$\begin{aligned} d\vec{s} \times \hat{r} &= R d\phi'(-\sin\phi'\hat{i} + \cos\phi'\hat{j}) \times [-R\cos\phi'\hat{i} + (y - R\sin\phi')\hat{j} + z\hat{k}] \\ &= R d\phi'[z\cos\phi'\hat{i} + z\sin\phi'\hat{j} + (R - y\sin\phi')\hat{k}] \end{aligned} \quad (9.8.3)$$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \vec{r}}{r^3} = \frac{\mu_0 IR}{4\pi} \frac{z \cos \phi' \hat{i} + z \sin \phi' \hat{j} + (R - y \sin \phi') \hat{k}}{(R^2 + y^2 + z^2 - 2yR \sin \phi')^{3/2}} d\phi' \quad (9.8.4)$$

Thus, magnetic field at P is

$$\vec{B}(0, y, z) = \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{z \cos \phi' \hat{i} + z \sin \phi' \hat{j} + (R - y \sin \phi') \hat{k}}{(R^2 + y^2 + z^2 - 2yR \sin \phi')^{3/2}} d\phi' \quad (9.8.5)$$

The x-component of $\vec{B}$ can be readily shown to be zero

$$B_x = \frac{\mu_0 IRz}{4\pi} \int_0^{2\pi} \frac{\cos \phi' d\phi'}{(R^2 + y^2 + z^2 - 2yR \sin \phi')^{3/2}} = 0 \quad (9.8.6)$$

by making a change of variable $w = R^2 + y^2 + z^2 - 2yR \sin \phi'$, followed by a straightforward integration. One may also invoke symmetry arguments to verify that B_x must vanish; namely, the contribution at ϕ' is cancelled by the contribution at $\pi - \phi'$. On the other hand, the y and the z components of $\vec{B}$,

$$B_y = \frac{\mu_0 IRz}{4\pi} \int_0^{2\pi} \frac{\sin \phi' d\phi'}{(R^2 + y^2 + z^2 - 2yR \sin \phi')^{3/2}} \quad (9.8.7)$$

and

$$B_z = \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{(R - y \sin \phi') d\phi'}{(R^2 + y^2 + z^2 - 2yR \sin \phi')^{3/2}} \quad (9.8.8)$$

involve *elliptic integrals* which can be evaluated numerically.

In the limit $y = 0$, the field point P is located along the z-axis, and we recover the results obtained in Example 9.2:

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \int_0^{2\pi} \sin \phi' d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2 + z^2)^{3/2}} \cos \phi' \Big|_0^{2\pi} = 0 \quad (9.8.9)$$

and

$$B_z = \frac{\mu_0}{4\pi} \frac{IR^2}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi IR^2}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2 + z^2)^{3/2}} \quad (9.8.10)$$

Now, let's consider the "point-dipole" limit where $R \ll (y^2 + z^2)^{1/2} = r$, i.e., the characteristic dimension of the current source is much smaller compared to the distance where the magnetic field is to be measured. In this limit, the denominator in the integrand can be expanded as

$$\begin{aligned} (R^2 + y^2 + z^2 - 2yR \sin \phi')^{-3/2} &= \frac{1}{r^3} \left[1 + \frac{R^2 - 2yR \sin \phi'}{r^2} \right]^{-3/2} \\ &= \frac{1}{r^3} \left[1 - \frac{3}{2} \left(\frac{R^2 - 2yR \sin \phi'}{r^2} \right) + \dots \right] \end{aligned} \quad (9.8.11)$$

This leads to

$$\begin{aligned} B_y &\approx \frac{\mu_0 I}{4\pi} \frac{Rz}{r^3} \int_0^{2\pi} \left[1 - \frac{3}{2} \left(\frac{R^2 - 2yR \sin \phi'}{r^2} \right) \right] \sin \phi' d\phi' \\ &= \frac{\mu_0 I}{4\pi} \frac{3R^2 yz}{r^5} \int_0^{2\pi} \sin^2 \phi' d\phi' = \frac{\mu_0 I}{4\pi} \frac{3\pi R^2 yz}{r^5} \end{aligned} \quad (9.8.12)$$

and

$$\begin{aligned} B_z &\approx \frac{\mu_0 I}{4\pi} \frac{R}{r^3} \int_0^{2\pi} \left[1 - \frac{3}{2} \left(\frac{R^2 - 2yR \sin \phi'}{r^2} \right) \right] (R - y \sin \phi') d\phi' \\ &= \frac{\mu_0 I}{4\pi} \frac{R}{r^3} \int_0^{2\pi} \left[\left(R - \frac{3R^3}{2r^2} \right) - \left(1 - \frac{9R^2}{2r^2} \right) \sin \phi' - \frac{3Ry^2}{r^2} \sin^2 \phi' \right] d\phi' \\ &= \frac{\mu_0 I}{4\pi} \frac{R}{r^3} \left[2\pi \left(R - \frac{3R^3}{2r^2} \right) - \frac{3\pi Ry^2}{r^2} \right] \\ &= \frac{\mu_0 I}{4\pi} \frac{\pi R^2}{r^3} \left[2 - \frac{3y^2}{r^2} + \text{higher order terms} \right] \end{aligned} \quad (9.8.13)$$

The quantity $I(\pi R^2)$ may be identified as the magnetic dipole moment $\mu = IA$, where $A = \pi R^2$ is the area of the loop. Using spherical coordinates where $y = r \sin \theta$ and $z = r \cos \theta$, the above expressions may be rewritten as

$$B_y = \frac{\mu_0 (I\pi R^2)}{4\pi} \frac{3(r \sin \theta)(r \cos \theta)}{r^5} = \frac{\mu_0}{4\pi} \frac{3\mu \sin \theta \cos \theta}{r^3} \quad (9.8.14)$$

and

$$B_z = \frac{\mu_0}{4\pi} \frac{(I\pi R^2)}{r^3} \left(2 - \frac{3r^2 \sin^2 \theta}{r^2} \right) = \frac{\mu_0}{4\pi} \frac{\mu}{r^3} (2 - 3\sin^2 \theta) = \frac{\mu_0}{4\pi} \frac{\mu}{r^3} (3\cos^2 \theta - 1) \quad (9.8.15)$$

Thus, we see that the magnetic field at a point $r \gg R$ due to a current ring of radius R may be approximated by a small magnetic dipole moment placed at the origin (Figure 9.8.2).

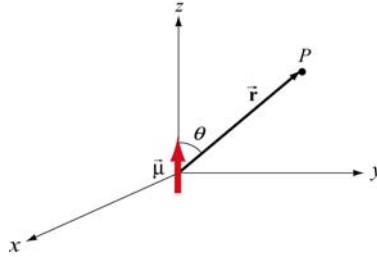


Figure 9.8.2 Magnetic dipole moment $\vec{\mu} = \mu \hat{k}$

The magnetic field lines due to a current loop and a dipole moment (small bar magnet) are depicted in Figure 9.8.3.

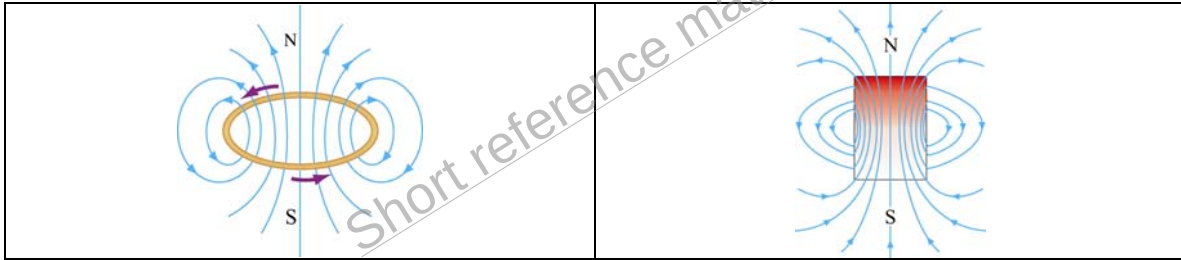


Figure 9.8.3 Magnetic field lines due to (a) a current loop, and (b) a small bar magnet.

The magnetic field at P can also be written in spherical coordinates

$$\vec{B} = B_r \hat{r} + B_\theta \hat{\theta} \quad (9.8.16)$$

The spherical components B_r and B_θ are related to the Cartesian components B_y and B_z by

$$B_r = B_y \sin \theta + B_z \cos \theta, \quad B_\theta = B_y \cos \theta - B_z \sin \theta \quad (9.8.17)$$

In addition, we have, for the unit vectors,

$$\hat{r} = \sin \theta \hat{j} + \cos \theta \hat{k}, \quad \hat{\theta} = \cos \theta \hat{j} - \sin \theta \hat{k} \quad (9.8.18)$$

Using the above relations, the spherical components may be written as

$$B_r = \frac{\mu_0 I R^2 \cos \theta}{4\pi} \int_0^{2\pi} \frac{d\phi'}{(R^2 + r^2 - 2rR \sin \theta \sin \phi')^{3/2}} \quad (9.8.19)$$

and

$$B_\theta(r, \theta) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{(r \sin \phi' - R \sin \theta) d\phi'}{(R^2 + r^2 - 2rR \sin \theta \sin \phi')^{3/2}} \quad (9.8.20)$$

In the limit where $R \ll r$, we obtain

$$B_r \approx \frac{\mu_0 I R^2 \cos \theta}{4\pi r^3} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi I R^2 \cos \theta}{r^3} = \frac{\mu_0}{4\pi} \frac{2\mu \cos \theta}{r^3} \quad (9.8.21)$$

and

$$\begin{aligned} B_\theta &= \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{(r \sin \phi' - R \sin \theta) d\phi'}{(R^2 + r^2 - 2rR \sin \theta \sin \phi')^{3/2}} \\ &\approx \frac{\mu_0 I R}{4\pi r^3} \int_0^{2\pi} \left[-R \sin \theta \left(1 - \frac{3R^2}{2r^2} \right) + \left(r - \frac{3R^2}{2r} - \frac{3R^2 \sin^2 \theta}{2r} \right) \sin \phi' + 3R \sin \theta \sin^2 \phi' \right] d\phi' \\ &\approx \frac{\mu_0 I R}{4\pi r^3} (-2\pi R \sin \theta + 3\pi R \sin \theta) = \frac{\mu_0 (I \pi R^2) \sin \theta}{4\pi r^3} \\ &= \frac{\mu_0}{4\pi} \frac{\mu \sin \theta}{r^3} \end{aligned} \quad (9.8.22)$$

9.9 Appendix 2: Helmholtz Coils

Consider two N -turn circular coils of radius R , each perpendicular to the axis of symmetry, with their centers located at $z = \pm l/2$. There is a steady current I flowing in the same direction around each coil, as shown in Figure 9.9.1. Let's find the magnetic field $\vec{B}$ on the axis at a distance z from the center of one coil.

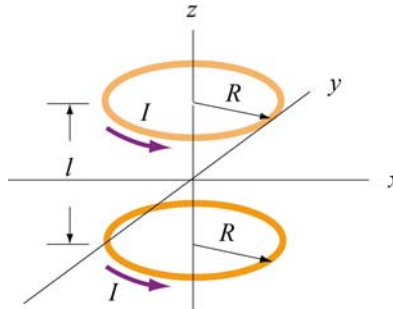


Figure 9.9.1 Helmholtz coils

Using the result shown in Example 9.2 for a single coil and applying the superposition principle, the magnetic field at $P(z,0)$ (a point at a distance $z-l/2$ away from one center and $z+l/2$ from the other) due to the two coils can be obtained as:

$$B_z = B_{\text{top}} + B_{\text{bottom}} = \frac{\mu_0 N I R^2}{2} \left[\frac{1}{[(z-l/2)^2 + R^2]^{3/2}} + \frac{1}{[(z+l/2)^2 + R^2]^{3/2}} \right] \quad (9.9.1)$$

A plot of B_z/B_0 with $B_0 = \frac{\mu_0 N I}{(5/4)^{3/2} R}$ being the field strength at $z=0$ and $l=R$ is depicted in Figure 9.9.2.

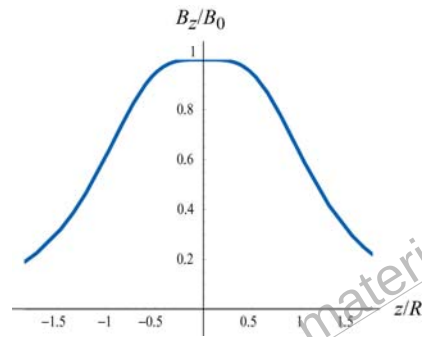


Figure 9.9.2 Magnetic field as a function of z/R .

Let's analyze the properties of B_z in more detail. Differentiating B_z with respect to z , we obtain

$$B'_z(z) = \frac{dB_z}{dz} = \frac{\mu_0 N I R^2}{2} \left\{ -\frac{3(z-l/2)}{[(z-l/2)^2 + R^2]^{5/2}} - \frac{3(z+l/2)}{[(z+l/2)^2 + R^2]^{5/2}} \right\} \quad (9.9.2)$$

One may readily show that at the midpoint, $z=0$, the derivative vanishes:

$$\left. \frac{dB}{dz} \right|_{z=0} = 0 \quad (9.9.3)$$

Straightforward differentiation yields

$$B''_z(z) = \frac{d^2 B}{dz^2} = \frac{N \mu_0 I R^2}{2} \left\{ -\frac{3}{[(z-l/2)^2 + R^2]^{5/2}} + \frac{15(z-l/2)^2}{[(z-l/2)^2 + R^2]^{7/2}} - \frac{3}{[(z+l/2)^2 + R^2]^{5/2}} + \frac{15(z+l/2)^2}{[(z+l/2)^2 + R^2]^{7/2}} \right\} \quad (9.9.4)$$

At the midpoint $z = 0$, the above expression simplifies to

$$\begin{aligned} B_z''(0) &= \left. \frac{d^2 B}{dz^2} \right|_{z=0} = \frac{\mu_0 N I^2}{2} \left\{ -\frac{6}{[(l/2)^2 + R^2]^{5/2}} + \frac{15l^2}{2[(l/2)^2 + R^2]^{7/2}} \right\} \\ &= -\frac{\mu_0 N I^2}{2} \frac{6(R^2 - l^2)}{[(l/2)^2 + R^2]^{7/2}} \end{aligned} \quad (9.9.5)$$

Thus, the condition that the second derivative of B_z vanishes at $z = 0$ is $l = R$. That is, the distance of separation between the two coils is equal to the radius of the coil. A configuration with $l = R$ is known as *Helmholtz coils*.

For small z , we may make a Taylor-series expansion of $B_z(z)$ about $z = 0$:

$$B_z(z) = B_z(0) + B_z'(0)z + \frac{1}{2!} B_z''(0)z^2 + \dots \quad (9.9.6)$$

The fact that the first two derivatives vanish at $z = 0$ indicates that the magnetic field is fairly uniform in the small z region. One may even show that the third derivative $B_z'''(0)$ vanishes at $z = 0$ as well.

Recall that the force experienced by a dipole in a magnetic field is $\vec{F}_B = \nabla(\vec{\mu} \cdot \vec{B})$. If we place a magnetic dipole $\vec{\mu} = \mu_z \hat{k}$ at $z = 0$, the magnetic force acting on the dipole is

$$\vec{F}_B = \nabla(\mu_z B_z) = \mu_z \left(\frac{dB_z}{dz} \right) \hat{k} \quad (9.9.7)$$

which is expected to be very small since the magnetic field is nearly uniform there.

Animation 9.5: Magnetic Field of the Helmholtz Coils

The animation in Figure 9.9.3(a) shows the magnetic field of the Helmholtz coils. In this configuration the currents in the top and bottom coils flow in the same direction, with their dipole moments aligned. The magnetic fields from the two coils add up to create a net field that is nearly uniform at the center of the coils. Since the distance between the coils is equal to the radius of the coils and remains unchanged, the force of attraction between them creates a tension, and is illustrated by field lines stretching out to enclose both coils. When the distance between the coils is not fixed, as in the animation depicted in Figure 9.9.3(b), the two coils move toward each other due to their force of attraction. In this animation, the top loop has only half the current as the bottom loop. The field configuration is shown using the “iron filings” representation.

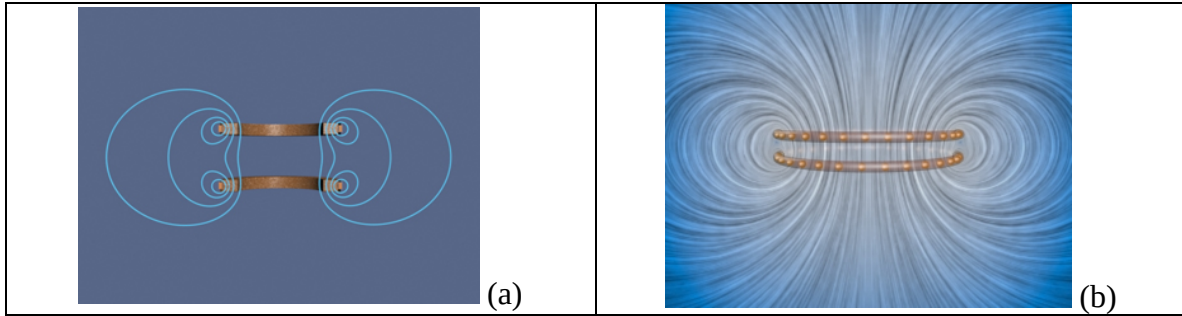


Figure 9.9.3 (a) Magnetic field of the Helmholtz coils where the distance between the coils is equal to the radius of the coil. (b) Two co-axial wire loops carrying current in the same sense are attracted to each other.

Next, let's consider the case where the currents in the loop flow in the opposite directions, as shown in Figure 9.9.4.

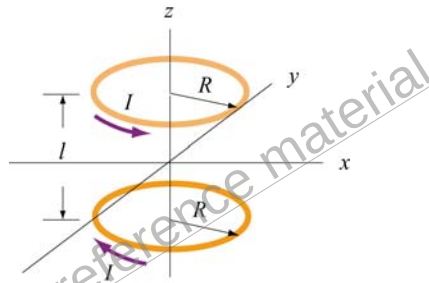


Figure 9.9.4 Two circular loops carrying currents in the opposite directions.

Again, by superposition principle, the magnetic field at a point $P(0, 0, z)$ with $z > 0$ is

$$B_z = B_{1z} + B_{2z} = \frac{\mu_0 N I R^2}{2} \left[\frac{1}{[(z - l/2)^2 + R^2]^{3/2}} - \frac{1}{[(z + l/2)^2 + R^2]^{3/2}} \right] \quad (9.9.8)$$

A plot of B_z / B_0 with $B_0 = \mu_0 N I / 2R$ and $l = R$ is depicted in Figure 9.9.5.

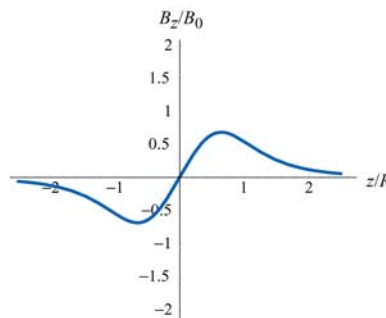


Figure 9.9.5 Magnetic field as a function of z / R .

Differentiating B_z with respect to z , we obtain

$$B'_z(z) = \frac{dB_z}{dz} = \frac{\mu_0 N I R^2}{2} \left\{ -\frac{3(z-l/2)}{[(z-l/2)^2 + R^2]^{5/2}} + \frac{3(z+l/2)}{[(z+l/2)^2 + R^2]^{5/2}} \right\} \quad (9.9.9)$$

At the midpoint, $z = 0$, we have

$$B'_z(0) = \left. \frac{dB_z}{dz} \right|_{z=0} = \frac{\mu_0 N I R^2}{2} \frac{3l}{[(l/2)^2 + R^2]^{5/2}} \neq 0 \quad (9.9.10)$$

Thus, a magnetic dipole $\vec{\mu} = \mu_z \hat{\mathbf{k}}$ placed at $z = 0$ will experience a net force:

$$\vec{\mathbf{F}}_B = \nabla(\vec{\mu} \cdot \vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z \left(\frac{dB_z(0)}{dz} \right) \hat{\mathbf{k}} = \frac{\mu_z \mu_0 N I R^2}{2} \frac{3l}{[(l/2)^2 + R^2]^{5/2}} \hat{\mathbf{k}} \quad (9.9.11)$$

For $l = R$, the above expression simplifies to

$$\vec{\mathbf{F}}_B = \frac{3\mu_z \mu_0 N I}{2(5/4)^{5/2} R^2} \hat{\mathbf{k}} \quad (9.9.12)$$

Animation 9.6: Magnetic Field of Two Coils Carrying Opposite Currents

The animation depicted in Figure 9.9.6 shows the magnetic field of two coils like the Helmholtz coils but with currents in the top and bottom coils flowing in the opposite directions. In this configuration, the magnetic dipole moments associated with each coil are anti-parallel.

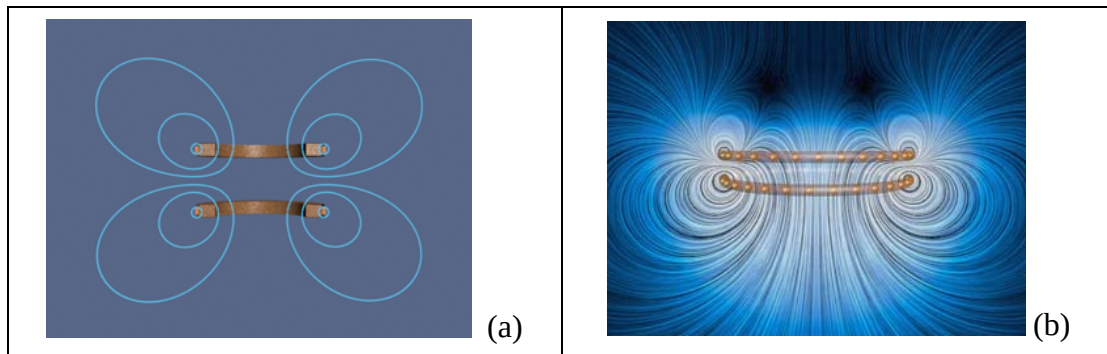


Figure 9.9.6 (a) Magnetic field due to coils carrying currents in the opposite directions. (b) Two co-axial wire loops carrying current in the opposite sense repel each other. The field configurations here are shown using the “iron filings” representation. The bottom wire loop carries twice the amount of current as the top wire loop.

At the center of the coils along the axis of symmetry, the magnetic field is zero. With the distance between the two coils fixed, the repulsive force results in a pressure between them. This is illustrated by field lines that are compressed along the central horizontal axis between the coils.

Animation 9.7: Forces Between Coaxial Current-Carrying Wires

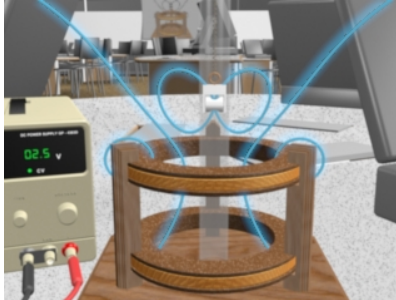


Figure 9.9.7 A magnet in the TeachSpin™ Magnetic Force apparatus when the current in the top coil is counterclockwise as seen from the top.

Figure 9.9.7 shows the force of repulsion between the magnetic field of a permanent magnet and the field of a current-carrying ring in the TeachSpin™ Magnetic Force apparatus. The magnet is forced to have its North magnetic pole pointing downward, and the current in the top coil of the Magnetic Force apparatus is moving clockwise as seen from above. The net result is a repulsion of the magnet when the current in this direction is increased. The visualization shows the stresses transmitted by the fields to the magnet when the current in the upper coil is increased.

Animation 9.8: Magnet Oscillating Between Two Coils

Figure 9.9.8 illustrates an animation in which the magnetic field of a permanent magnet suspended by a spring in the TeachSpin™ apparatus (see TeachSpin visualization), plus the magnetic field due to current in the two coils (here we see a "cutaway" cross-section of the apparatus).

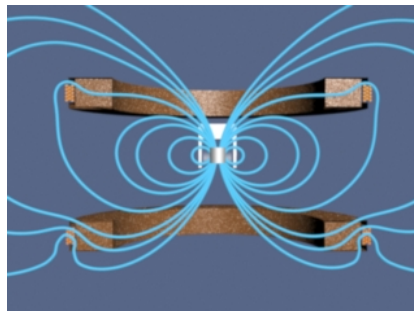


Figure 9.9.8 Magnet oscillating between two coils

The magnet is fixed so that its north pole points upward, and the current in the two coils is sinusoidal and 180 degrees out of phase. When the effective dipole moment of the top coil points upwards, the dipole moment of the bottom coil points downwards. Thus, the magnet is attracted to the upper coil and repelled by the lower coil, causing it to move upwards. When the conditions are reversed during the second half of the cycle, the magnet moves downwards.

This process can also be described in terms of tension along, and pressure perpendicular to, the field lines of the resulting field. When the dipole moment of one of the coils is aligned with that of the magnet, there is a tension along the field lines as they attempt to "connect" the coil and magnet. Conversely, when their moments are anti-aligned, there is a pressure perpendicular to the field lines as they try to keep the coil and magnet apart.

Animation 9.9: Magnet Suspended Between Two Coils

Figure 9.9.9 illustrates an animation in which the magnetic field of a permanent magnet suspended by a spring in the *TeachSpin*TM apparatus (see *TeachSpin* visualization), plus the magnetic field due to current in the two coils (here we see a "cutaway" cross-section of the apparatus). The magnet is fixed so that its north pole points upward, and the current in the two coils is sinusoidal and in phase. When the effective dipole moment of the top coil points upwards, the dipole moment of the bottom coil points upwards as well. Thus, the magnet the magnet is attracted to both coils, and as a result feels no net force (although it does feel a torque, not shown here since the direction of the magnet is fixed to point upwards). When the dipole moments are reversed during the second half of the cycle, the magnet is repelled by both coils, again resulting in no net force.

This process can also be described in terms of tension along, and pressure perpendicular to, the field lines of the resulting field. When the dipole moment of the coils is aligned with that of the magnet, there is a tension along the field lines as they are "pulled" from both sides. Conversely, when their moments are anti-aligned, there is a pressure perpendicular to the field lines as they are "squeezed" from both sides.

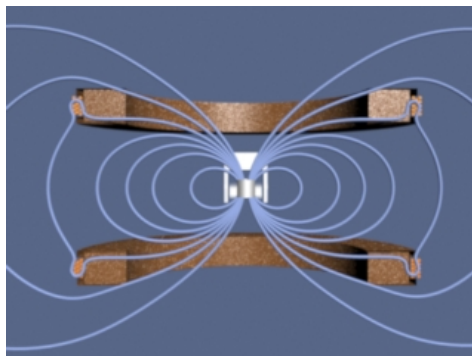


Figure 9.9.9 Magnet suspended between two coils

9.10 Problem-Solving Strategies

In this Chapter, we have seen how Biot-Savart and Ampere's laws can be used to calculate magnetic field due to a current source.

9.10.1 Biot-Savart Law:

The law states that the magnetic field at a point P due to a length element $d\vec{s}$ carrying a steady current I located at $\vec{r}$ away is given by

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \vec{r}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I d\vec{s}$, and the vector $\vec{r}'$ describing the position of $I d\vec{s}$. The magnitude $r' = |\vec{r}'|$ is the distance between $I d\vec{s}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point P is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{r}_p$ for the field point P . The quantity $r_p = |\vec{r}_p|$ is the distance between the origin and P .

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{r} = \vec{r}_p - \vec{r}'$. The corresponding unit vector is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r}_p - \vec{r}'}{|\vec{r}_p - \vec{r}'|}$$

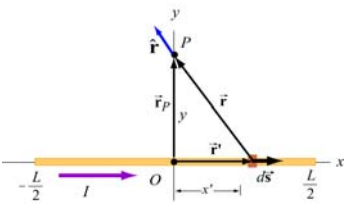
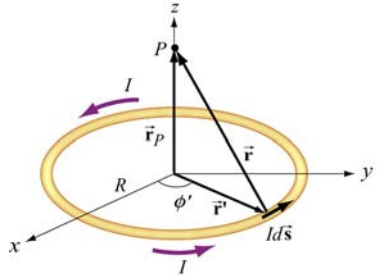
where $r = |\vec{r}| = |\vec{r}_p - \vec{r}'|$ is the distance between the source and the field point P .

(4) Calculate the cross product $d\vec{s} \times \hat{r}$ or $d\vec{s} \times \vec{r}$. The resultant vector gives the direction of the magnetic field $\vec{B}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained to $d\vec{B}$ and simplify as much as possible.

(6) Complete the integration to obtain B if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

Below we illustrate how these steps are executed for a current-carrying wire of length L and a loop of radius R .

Current distribution	Finite wire of length L	Circular loop of radius R
Figure		
(1) Source point	$\vec{r}' = x' \hat{i}$ $d\vec{s} = (d\vec{r}'/dx')dx' = dx' \hat{i}$	$\vec{r}' = R(\cos \phi' \hat{i} + \sin \phi' \hat{j})$ $d\vec{s} = (d\vec{r}'/d\phi')d\phi' = R d\phi'(-\sin \phi' \hat{i} + \cos \phi' \hat{j})$
(2) Field point P	$\vec{r}_p = y \hat{j}$	$\vec{r}_p = z \hat{k}$
(3) Relative position vector $\vec{r} = \vec{r}_p - \vec{r}'$	$\vec{r} = y \hat{j} - x' \hat{i}$ $r = \vec{r} = \sqrt{x'^2 + y^2}$ $\hat{r} = \frac{y \hat{j} - x' \hat{i}}{\sqrt{x'^2 + y^2}}$	$\vec{r} = -R \cos \phi' \hat{i} - R \sin \phi' \hat{j} + z \hat{k}$ $r = \vec{r} = \sqrt{R^2 + z^2}$ $\hat{r} = \frac{-R \cos \phi' \hat{i} - R \sin \phi' \hat{j} + z \hat{k}}{\sqrt{R^2 + z^2}}$
(4) The cross product $d\vec{s} \times \hat{r}$	$d\vec{s} \times \hat{r} = \frac{y dx' \hat{k}}{\sqrt{y^2 + x'^2}}$	$d\vec{s} \times \hat{r} = \frac{R d\phi' (z \cos \phi' \hat{i} + z \sin \phi' \hat{j} + R \hat{k})}{\sqrt{R^2 + z^2}}$
(5) Rewrite $d\vec{B}$	$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{y dx' \hat{k}}{(y^2 + x'^2)^{3/2}}$	$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{R d\phi' (z \cos \phi' \hat{i} + z \sin \phi' \hat{j} + R \hat{k})}{(R^2 + z^2)^{3/2}}$
(6) Integrate to get $\vec{B}$	$B_x = 0$ $B_y = 0$ $B_z = \frac{\mu_0 I y}{4\pi} \int_{-L/2}^{L/2} \frac{dx'}{(y^2 + x'^2)^{3/2}}$ $= \frac{\mu_0 I}{4\pi} \frac{L}{y \sqrt{y^2 + (L/2)^2}}$	$B_x = \frac{\mu_0 I R z}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} \cos \phi' d\phi' = 0$ $B_y = \frac{\mu_0 I R z}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} \sin \phi' d\phi' = 0$ $B_z = \frac{\mu_0 I R^2}{4\pi (R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}$

9.10.2 Ampere's law:

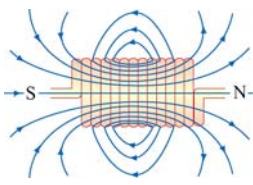
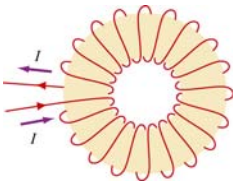
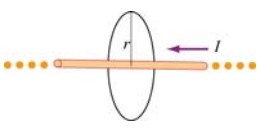
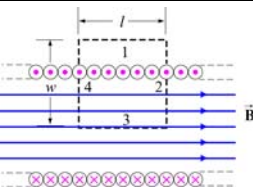
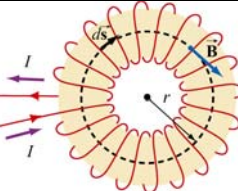
Ampere's law states that the line integral of $\vec{B} \cdot d\vec{s}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

- (1) Draw an Amperian loop using symmetry arguments.
- (2) Find the current enclosed by the Amperian loop.
- (3) Calculate the line integral $\oint \vec{B} \cdot d\vec{s}$ around the closed loop.
- (4) Equate $\oint \vec{B} \cdot d\vec{s}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{B}$.

Below we summarize how the methodology can be applied to calculate the magnetic field for an infinite wire, an ideal solenoid and a toroid.

System	Infinite wire	Ideal solenoid	Toroid
Figure			
(1) Draw the Amperian loop			
(2) Find the current enclosed by the Amperian loop	$I_{\text{enc}} = I$	$I_{\text{enc}} = NI$	$I_{\text{enc}} = NI$
(3) Calculate $\oint \vec{B} \cdot d\vec{s}$ along the loop	$\oint \vec{B} \cdot d\vec{s} = B(2\pi r)$	$\oint \vec{B} \cdot d\vec{s} = Bl$	$\oint \vec{B} \cdot d\vec{s} = B(2\pi r)$

(4) Equate $\mu_0 I_{\text{enc}}$ with $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ to obtain $\vec{\mathbf{B}}$	$B = \frac{\mu_0 I}{2\pi r}$	$B = \frac{\mu_0 NI}{l} = \mu_0 nI$	$B = \frac{\mu_0 NI}{2\pi r}$
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9.11 Solved Problems

9.11.1 Magnetic Field of a Straight Wire

Consider a straight wire of length L carrying a current I along the $+x$ -direction, as shown in Figure 9.11.1 (ignore the return path of the current or the source for the current.) What is the magnetic field at an arbitrary point P on the xy -plane?

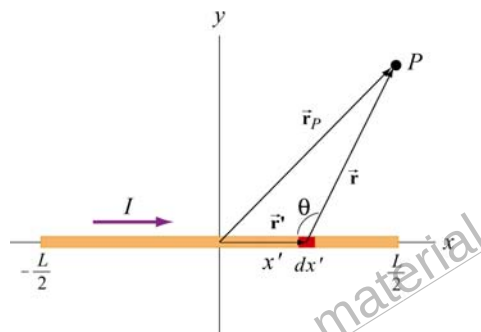


Figure 9.11.1 A finite straight wire carrying a current I .

Solution:

The problem is very similar to Example 9.1. However, now the field point is an arbitrary point in the xy -plane. Once again we solve the problem using the methodology outlined in Section 9.10.

(1) Source point

From Figure 9.10.1, we see that the infinitesimal length dx' described by the position vector $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$ constitutes a current source $I d\vec{\mathbf{s}} = (Idx')\hat{\mathbf{i}}$.

(2) Field point

As can be seen from Figure 9.10.1, the position vector for the field point P is $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

(3) Relative position vector

The relative position vector from the source to P is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_p - \vec{\mathbf{r}}' = (x - x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, with $r = |\vec{\mathbf{r}}_p| = |\vec{\mathbf{r}} - \vec{\mathbf{r}}'| = [(x - x')^2 + y^2]^{1/2}$ being the distance. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_p - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_p - \vec{\mathbf{r}}'|} = \frac{(x-x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{[(x-x')^2 + y^2]^{1/2}}$$

(4) Simplifying the cross product

The cross product $d\vec{\mathbf{s}} \times \vec{\mathbf{r}}$ can be simplified as

$$(dx'\hat{\mathbf{i}}) \times [(x-x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}] = y dx' \hat{\mathbf{k}}$$

where we have used $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$ and $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

(5) Writing down $d\vec{\mathbf{B}}$

Using the Biot-Savart law, the infinitesimal contribution due to $Id\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\mathbf{s}} \times \vec{\mathbf{r}}}{r^3} = \frac{\mu_0 I}{4\pi} \frac{y dx'}{[(x-x')^2 + y^2]^{3/2}} \hat{\mathbf{k}} \quad (9.11.1)$$

Thus, we see that the direction of the magnetic field is in the $+\hat{\mathbf{k}}$ direction.

(6) Carrying out the integration to obtain $\vec{\mathbf{B}}$

The total magnetic field at P can then be obtained by integrating over the entire length of the wire:

$$\begin{aligned} \vec{\mathbf{B}} &= \int_{\text{wire}} d\vec{\mathbf{B}} = \int_{-L/2}^{L/2} \frac{\mu_0 I y dx'}{4\pi [(x-x')^2 + y^2]^{3/2}} \hat{\mathbf{k}} = -\frac{\mu_0 I}{4\pi y} \frac{(x-x')}{\sqrt{(x-x')^2 + y^2}} \bigg|_{-L/2}^{L/2} \hat{\mathbf{k}} \\ &= -\frac{\mu_0 I}{4\pi y} \left[\frac{(x-L/2)}{\sqrt{(x-L/2)^2 + y^2}} - \frac{(x+L/2)}{\sqrt{(x+L/2)^2 + y^2}} \right] \hat{\mathbf{k}} \end{aligned} \quad (9.11.2)$$

Let's consider the following limits:

(i) $x = 0$

In this case, the field point P is at $(x, y) = (0, y)$ on the y axis. The magnetic field becomes

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{4\pi y} \left[\frac{-L/2}{\sqrt{(-L/2)^2 + y^2}} - \frac{+L/2}{\sqrt{(+L/2)^2 + y^2}} \right] \hat{\mathbf{k}} = \frac{\mu_0 I}{2\pi y} \frac{L/2}{\sqrt{(L/2)^2 + y^2}} \hat{\mathbf{k}} = \frac{\mu_0 I}{2\pi y} \cos \theta \hat{\mathbf{k}} \quad (9.11.3)$$

in agreement with Eq. (9.1.6).

(ii) Infinite length limit

Consider the limit where $L \gg x, y$. This gives back the expected infinite-length result:

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{4\pi y} \left[\frac{-L/2}{L/2} - \frac{+L/2}{L/2} \right] \hat{\mathbf{k}} = \frac{\mu_0 I}{2\pi y} \hat{\mathbf{k}} \quad (9.11.4)$$

If we use cylindrical coordinates with the wire pointing along the $+z$ -axis then the magnetic field is given by the expression

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{\phi}} \quad (9.11.5)$$

where $\hat{\boldsymbol{\phi}}$ is the tangential unit vector and the field point P is a distance r away from the wire.

9.11.2 Current-Carrying Arc

Consider the current-carrying loop formed of radial lines and segments of circles whose centers are at point P as shown below. Find the magnetic field $\vec{\mathbf{B}}$ at P .

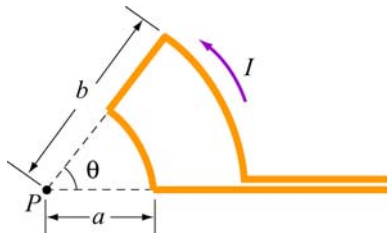


Figure 9.11.2 Current-carrying arc

Solution:

According to the Biot-Savart law, the magnitude of the magnetic field due to a differential current-carrying element $I d\vec{s}$ is given by

$$dB = \frac{\mu_0 I}{4\pi} \frac{|d\vec{s} \times \hat{r}|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{r d\theta'}{r^2} = \frac{\mu_0 I}{4\pi r} d\theta' \quad (9.11.6)$$

For the outer arc, we have

$$B_{\text{outer}} = \frac{\mu_0 I}{4\pi b} \int_0^\theta d\theta' = \frac{\mu_0 I \theta}{4\pi b} \quad (9.11.7)$$

The direction of $\vec{B}_{\text{outer}}$ is determined by the cross product $d\vec{s} \times \hat{r}$ which points out of the page. Similarly, for the inner arc, we have

$$B_{\text{inner}} = \frac{\mu_0 I}{4\pi a} \int_0^\theta d\theta' = \frac{\mu_0 I \theta}{4\pi a} \quad (9.11.8)$$

For $\vec{B}_{\text{inner}}$, $d\vec{s} \times \hat{r}$ points into the page. Thus, the total magnitude of magnetic field is

$$\vec{B} = \vec{B}_{\text{inner}} + \vec{B}_{\text{outer}} = \frac{\mu_0 I \theta}{4\pi} \left(\frac{1}{a} - \frac{1}{b} \right) \text{ (into page)} \quad (9.11.9)$$

9.11.3 Rectangular Current Loop

Determine the magnetic field (in terms of I , a and b) at the origin O due to the current loop shown in Figure 9.11.3

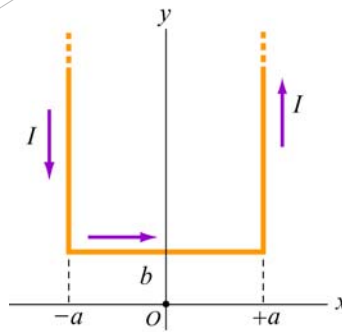


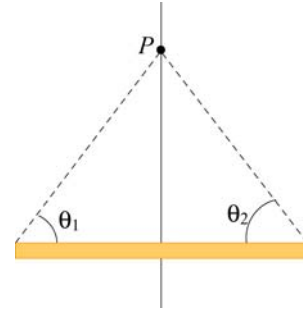
Figure 9.11.3 Rectangular current loop

Solution:

For a finite wire carrying a current I , the contribution to the magnetic field at a point P is given by Eq. (9.1.5):

$$B = \frac{\mu_0 I}{4\pi r} (\cos \theta_1 + \cos \theta_2)$$

where θ_1 and θ_2 are the angles which parameterize the length of the wire.



To obtain the magnetic field at O , we make use of the above formula. The contributions can be divided into three parts:

(i) Consider the left segment of the wire which extends from $(x, y) = (-a, +\infty)$ to $(-a, +d)$. The angles which parameterize this segment give $\cos \theta_1 = 1$ ($\theta_1 = 0$) and $\cos \theta_2 = -b / \sqrt{b^2 + a^2}$. Therefore,

$$B_1 = \frac{\mu_0 I}{4\pi a} (\cos \theta_1 + \cos \theta_2) = \frac{\mu_0 I}{4\pi a} \left(1 - \frac{b}{\sqrt{b^2 + a^2}} \right) \quad (9.11.10)$$

The direction of $\vec{B}_1$ is out of page, or $+\hat{k}$.

(ii) Next, we consider the segment which extends from $(x, y) = (-a, +b)$ to $(+a, +b)$. Again, the (cosine of the) angles are given by

$$\cos \theta_1 = \frac{a}{\sqrt{a^2 + b^2}} \quad (9.11.11)$$

$$\cos \theta_2 = \cos \theta_1 = \frac{a}{\sqrt{a^2 + b^2}} \quad (9.11.12)$$

This leads to

$$B_2 = \frac{\mu_0 I}{4\pi b} \left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \right) = \frac{\mu_0 I a}{2\pi b \sqrt{a^2 + b^2}} \quad (9.11.13)$$

The direction of $\vec{B}_2$ is into the page, or $-\hat{k}$.

(iii) The third segment of the wire runs from $(x, y) = (+a, +b)$ to $(+a, +\infty)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 \quad (9.11.14)$$

The direction of $\vec{\mathbf{B}}_3$ is again out of page, or $+\hat{\mathbf{k}}$.

The magnetic field is

$$\begin{aligned}\vec{\mathbf{B}} &= \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I}{2\pi a} \left(1 - \frac{b}{\sqrt{a^2 + b^2}} \right) \hat{\mathbf{k}} - \frac{\mu_0 I a}{2\pi b \sqrt{a^2 + b^2}} \hat{\mathbf{k}} \\ &= \frac{\mu_0 I}{2\pi a b \sqrt{a^2 + b^2}} \left(b\sqrt{a^2 + b^2} - b^2 - a^2 \right) \hat{\mathbf{k}}\end{aligned}\quad (9.11.15)$$

Note that in the limit $a \rightarrow 0$, the horizontal segment is absent, and the two semi-infinite wires carrying currents in the opposite direction overlap each other and their contributions completely cancel. Thus, the magnetic field vanishes in this limit.

9.11.4 Hairpin-Shaped Current-Carrying Wire

An infinitely long current-carrying wire is bent into a hairpin-like shape shown in Figure 9.11.4. Find the magnetic field at the point P which lies at the center of the half-circle.

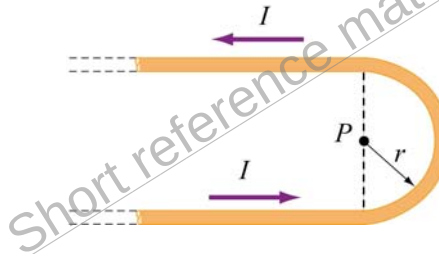


Figure 9.11.4 Hairpin-shaped current-carrying wire

Solution:

Again we break the wire into three parts: two semi-infinite plus a semi-circular segments.

(i) Let P be located at the origin in the xy plane. The first semi-infinite segment then extends from $(x, y) = (-\infty, -r)$ to $(0, -r)$. The two angles which parameterize this segment are characterized by $\cos \theta_1 = 1$ ($\theta_1 = 0$) and $\cos \theta_2 = 0$ ($\theta_2 = \pi/2$). Therefore, its contribution to the magnetic field at P is

$$B_1 = \frac{\mu_0 I}{4\pi r} (\cos \theta_1 + \cos \theta_2) = \frac{\mu_0 I}{4\pi r} (1 + 0) = \frac{\mu_0 I}{4\pi r} \quad (9.11.16)$$

The direction of $\vec{\mathbf{B}}_1$ is out of page, or $+\hat{\mathbf{k}}$.

(ii) For the semi-circular arc of radius r , we make use of the Biot-Savart law:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2} \quad (9.11.17)$$

and obtain

$$B_2 = \frac{\mu_0 I}{4\pi} \int_0^\pi \frac{rd\theta}{r^2} = \frac{\mu_0 I}{4r} \quad (9.11.18)$$

The direction of $\vec{B}_2$ is out of page, or $+\hat{k}$.

(iii) The third segment of the wire runs from $(x, y) = (0, +r)$ to $(-\infty, +r)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 = \frac{\mu_0 I}{4\pi r} \quad (9.11.19)$$

The direction of $\vec{B}_3$ is again out of page, or $+\hat{k}$.

The total magnitude of the magnetic field is

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 = 2\vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I}{2\pi r} \hat{k} + \frac{\mu_0 I}{4r} \hat{k} = \frac{\mu_0 I}{4\pi r} (2 + \pi) \hat{k} \quad (9.11.20)$$

Notice that the contribution from the two semi-infinite wires is equal to that due to an infinite wire:

$$\vec{B}_1 + \vec{B}_3 = 2\vec{B}_1 = \frac{\mu_0 I}{2\pi r} \hat{k} \quad (9.11.21)$$

9.11.5 Two Infinitely Long Wires

Consider two infinitely long wires carrying currents are in the $-x$ -direction.

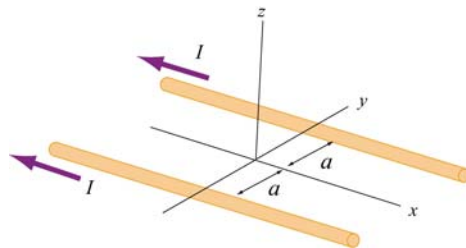


Figure 9.11.5 Two infinitely long wires

- (a) Plot the magnetic field pattern in the yz -plane.
- (b) Find the distance d along the z -axis where the magnetic field is a maximum.

Solutions:

- (a) The magnetic field lines are shown in Figure 9.11.6. Notice that the directions of both currents are into the page.

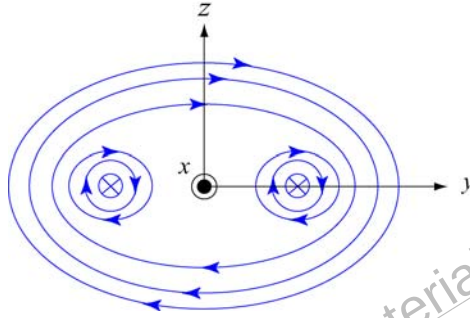


Figure 9.11.6 Magnetic field lines of two wires carrying current in the same direction.

- (b) The magnetic field at $(0, 0, z)$ due to wire 1 on the left is, using Ampere's law:

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi \sqrt{a^2 + z^2}} \quad (9.11.22)$$

Since the current is flowing in the $-x$ -direction, the magnetic field points in the direction of the cross product

$$(-\hat{i}) \times \hat{r}_1 = (-\hat{i}) \times (\cos \theta \hat{j} + \sin \theta \hat{k}) = \sin \theta \hat{j} - \cos \theta \hat{k} \quad (9.11.23)$$

Thus, we have

$$\vec{B}_1 = \frac{\mu_0 I}{2\pi \sqrt{a^2 + z^2}} (\sin \theta \hat{j} - \cos \theta \hat{k}) \quad (9.11.24)$$

For wire 2 on the right, the magnetic field strength is the same as the left one: $B_1 = B_2$. However, its direction is given by

$$(-\hat{i}) \times \hat{r}_2 = (-\hat{i}) \times (-\cos \theta \hat{j} + \sin \theta \hat{k}) = \sin \theta \hat{j} + \cos \theta \hat{k} \quad (9.11.25)$$

Adding up the contributions from both wires, the z -components cancel (as required by symmetry), and we arrive at

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 I \sin \theta}{\pi \sqrt{a^2 + z^2}} \hat{j} = \frac{\mu_0 I z}{\pi (a^2 + z^2)} \hat{j} \quad (9.11.26)$$

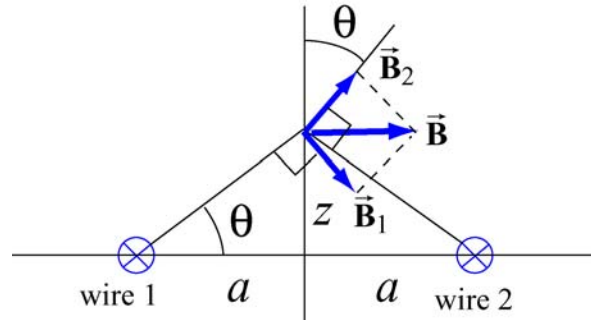


Figure 9.11.7 Superposition of magnetic fields due to two current sources

To locate the maximum of B , we set $dB/dz = 0$ and find

$$\frac{dB}{dz} = \frac{\mu_0 I}{\pi} \left(\frac{1}{a^2 + z^2} - \frac{2z^2}{(a^2 + z^2)^2} \right) = \frac{\mu_0 I}{\pi} \frac{a^2 - z^2}{(a^2 + z^2)^2} = 0 \quad (9.11.27)$$

which gives

$$z = a \quad (9.11.28)$$

Thus, at $z=a$, the magnetic field strength is a maximum, with a magnitude

$$B_{\max} = \frac{\mu_0 I}{2\pi a} \quad (9.11.29)$$

9.11.6 Non-Uniform Current Density

Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a non-uniform current density

$$J = \alpha r \quad (9.11.30)$$

where α is a constant. Find the magnetic field everywhere.

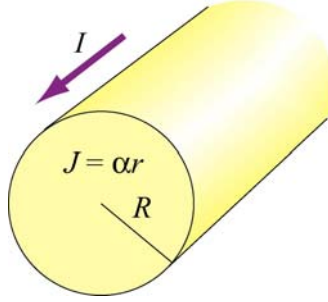


Figure 9.11.8 Non-uniform current density

Solution:

The problem can be solved by using the Ampere's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} \quad (9.11.31)$$

where the enclosed current I_{enc} is given by

$$I_{\text{enc}} = \int \vec{J} \cdot d\vec{A} = \int (\alpha r') (2\pi r' dr') \quad (9.11.32)$$

(a) For $r < R$, the enclosed current is

$$I_{\text{enc}} = \int_0^r 2\pi \alpha r'^2 dr' = \frac{2\pi \alpha r^3}{3} \quad (9.11.33)$$

Applying Ampere's law, the magnetic field at P_1 is given by

$$B_1 (2\pi r) = \frac{2\mu_0 \pi \alpha r^3}{3} \quad (9.11.34)$$

or

$$\boxed{B_1 = \frac{\alpha \mu_0}{3} r^2} \quad (9.11.35)$$

The direction of the magnetic field $\vec{B}_1$ is tangential to the Amperian loop which encloses the current.

(b) For $r > R$, the enclosed current is

$$I_{\text{enc}} = \int_0^R 2\pi \alpha r'^2 dr' = \frac{2\pi \alpha R^3}{3} \quad (9.11.36)$$

which yields

$$B_2(2\pi r) = \frac{2\mu_0\pi\alpha R^3}{3} \quad (9.11.37)$$

Thus, the magnetic field at a point P_2 outside the conductor is

$$B_2 = \frac{\alpha\mu_0 R^3}{3r} \quad (9.11.38)$$

A plot of B as a function of r is shown in Figure 9.11.9:

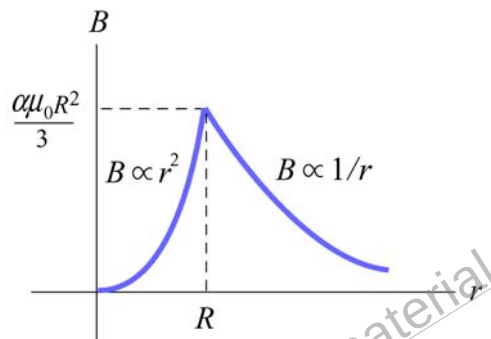


Figure 9.11.9 The magnetic field as a function of distance away from the conductor

9.11.7 Thin Strip of Metal

Consider an infinitely long, thin strip of metal of width w lying in the xy plane. The strip carries a current I along the $+x$ -direction, as shown in Figure 9.11.10. Find the magnetic field at a point P which is in the plane of the strip and at a distance s away from it.

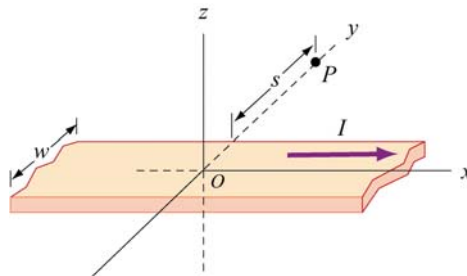


Figure 9.11.10 Thin strip of metal

Solution:

Consider a thin strip of width dr parallel to the direction of the current and at a distance r away from P , as shown in Figure 9.11.11. The amount of current carried by this differential element is

$$dI = I \left(\frac{dr}{w} \right) \quad (9.11.39)$$

Using Ampere's law, we see that the strip's contribution to the magnetic field at P is given by

$$dB(2\pi r) = \mu_0 I_{\text{enc}} = \mu_0 (dI) \quad (9.11.40)$$

or

$$dB = \frac{\mu_0 dI}{2\pi r} = \frac{\mu_0}{2\pi r} \left(\frac{I dr}{w} \right) \quad (9.11.41)$$

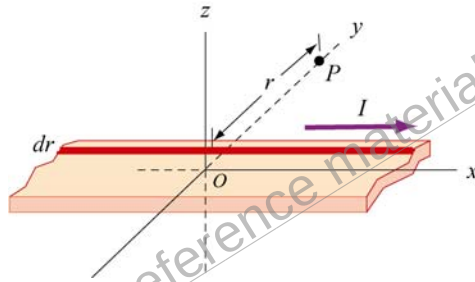


Figure 9.11.11 A thin strip with thickness dr carrying a steady current I .

Integrating this expression, we obtain

$$B = \int_s^{s+w} \frac{\mu_0 I}{2\pi w} \left(\frac{dr}{r} \right) = \frac{\mu_0 I}{2\pi w} \ln \left(\frac{s+w}{s} \right) \quad (9.11.42)$$

Using the right-hand rule, the direction of the magnetic field can be shown to point in the +z-direction, or

$$\vec{B} = \frac{\mu_0 I}{2\pi w} \ln \left(1 + \frac{w}{s} \right) \hat{k} \quad (9.11.43)$$

Notice that in the limit of vanishing width, $w \ll s$, $\ln(1 + w/s) \approx w/s$, and the above expression becomes

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{k} \quad (9.11.44)$$

which is the magnetic field due to an infinitely long thin straight wire.

9.11.8 Two Semi-Infinite Wires

A wire carrying current I runs down the y axis to the origin, thence out to infinity along the positive x axis. Show that the magnetic field in the quadrant with $x, y > 0$ of the xy plane is given by

$$B_z = \frac{\mu_0 I}{4\pi} \left(\frac{1}{x} + \frac{1}{y} + \frac{x}{y\sqrt{x^2 + y^2}} + \frac{y}{x\sqrt{x^2 + y^2}} \right) \quad (9.11.45)$$

Solution:

Let $P(x, y)$ be a point in the first quadrant at a distance r_1 from a point $(0, y')$ on the y -axis and distance r_2 from $(x', 0)$ on the x -axis.

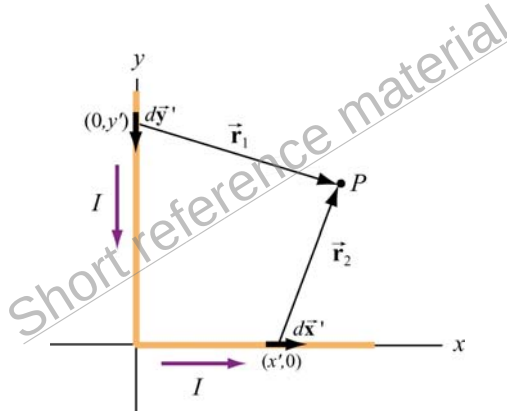


Figure 9.11.12 Two semi-infinite wires

Using the Biot-Savart law, the magnetic field at P is given by

$$\vec{B} = \int d\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \int_{\text{wire } y} \frac{d\vec{s}_1 \times \hat{r}_1}{r_1^2} + \frac{\mu_0 I}{4\pi} \int_{\text{wire } x} \frac{d\vec{s}_2 \times \hat{r}_2}{r_2^2} \quad (9.11.46)$$

Let's analyze each segment separately.

(i) Along the y axis, consider a differential element $d\vec{s}_1 = -dy' \hat{j}$ which is located at a distance $\vec{r}_1 = x\hat{i} + (y - y')\hat{j}$ from P . This yields

$$d\vec{s}_1 \times \vec{r}_1 = (-dy' \hat{j}) \times [x\hat{i} + (y - y')\hat{j}] = x dy' \hat{k} \quad (9.11.47)$$

(ii) Similarly, along the x -axis, we have $d\vec{s}_2 = dx'\hat{i}$ and $\vec{r}_2 = (x - x')\hat{i} + y\hat{j}$ which gives

$$d\vec{s}_2 \times \vec{r}_2 = y dx' \hat{k} \quad (9.11.48)$$

Thus, we see that the magnetic field at P points in the $+z$ -direction. Using the above results and $r_1 = \sqrt{x^2 + (y - y')^2}$ and $r_2 = \sqrt{(x - x')^2 + y^2}$, we obtain

$$B_z = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{x dy'}{[x^2 + (y - y')^2]^{3/2}} + \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{y dx'}{[y^2 + (x - x')^2]^{3/2}} \quad (9.11.49)$$

The integrals can be readily evaluated using

$$\int_0^\infty \frac{b ds}{[b^2 + (a - s)^2]^{3/2}} = \frac{1}{b} + \frac{a}{b\sqrt{a^2 + b^2}} \quad (9.11.50)$$

The final expression for the magnetic field is given by

$$\vec{B} = \frac{\mu_0 I}{4\pi} \left[\frac{1}{x} + \frac{y}{x\sqrt{x^2 + y^2}} + \frac{1}{y} + \frac{x}{y\sqrt{x^2 + y^2}} \right] \hat{k} \quad (9.11.51)$$

We may show that the result is consistent with Eq. (9.1.5)

9.12 Conceptual Questions

1. Compare and contrast Biot-Savart law in magnetostatics with Coulomb's law in electrostatics.
2. If a current is passed through a spring, does the spring stretch or compress? Explain.
3. How is the path of the integration of $\oint \vec{B} \cdot d\vec{s}$ chosen when applying Ampere's law?
4. Two concentric, coplanar circular loops of different diameters carry steady currents in the same direction. Do the loops attract or repel each other? Explain.
5. Suppose three infinitely long parallel wires are arranged in such a way that when looking at the cross section, they are at the corners of an equilateral triangle. Can currents be arranged (combination of flowing in or out of the page) so that all three wires (a) attract, and (b) repel each other? Explain.

9.13 Additional Problems

9.13.1 Application of Ampere's Law

The simplest possible application of Ampere's law allows us to calculate the magnetic field in the vicinity of a single infinitely long wire. Adding more wires with differing currents will check your understanding of Ampere's law.

(a) Calculate with Ampere's law the magnetic field, $|\vec{B}| = B(r)$, as a function of distance r from the wire, in the vicinity of an infinitely long straight wire that carries current I . Show with a sketch the integration path you choose and state explicitly how you use symmetry. What is the field at a distance of 10 mm from the wire if the current is 10 A?

(b) Eight parallel wires cut the page perpendicularly at the points shown. A wire labeled with the integer k ($k = 1, 2, \dots, 8$) bears the current $2k$ times I_0 (i.e., $I_k = 2k I_0$). For those with $k = 1$ to 4, the current flows up out of the page; for the rest, the current flows down into the page. Evaluate $\oint \vec{B} \cdot d\vec{s}$ along the closed path (see figure) in the direction indicated by the arrowhead. (Watch your signs!)

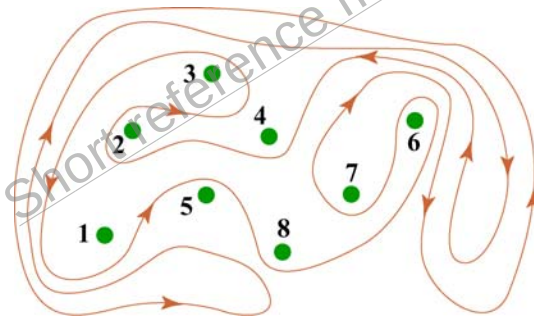


Figure 9.13.1 Amperian loop

(c) Can you use a single application of Ampere's Law to find the field at a point in the vicinity of the 8 wires? Why? How would you proceed to find the field at an arbitrary point P ?

9.13.2 Magnetic Field of a Current Distribution from Ampere's Law

Consider the cylindrical conductor with a hollow center and copper walls of thickness $b - a$ as shown in Figure 9.13.2. The radii of the inner and outer walls are a and b respectively, and the current I is uniformly spread over the cross section of the copper.

- (a) Calculate the magnitude of the magnetic field in the region outside the conductor, $r > b$. (Hint: consider the entire conductor to be a single thin wire, construct an Amperian loop, and apply Ampere's Law.) What is the direction of $\vec{B}$?

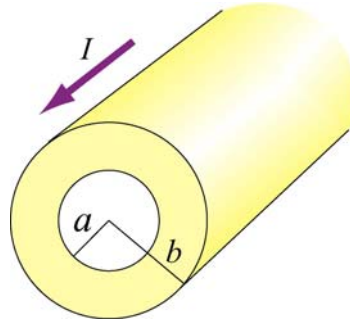


Figure 9.13.2 Hollow cylinder carrying a steady current I .

- (b) Calculate the magnetic field inside the inner radius, $r < a$. What is the direction of $\vec{B}$?
- (c) Calculate the magnetic field within the inner conductor, $a < r < b$. What is the direction of $\vec{B}$?
- (d) Plot the behavior of the magnitude of the magnetic field $B(r)$ from $r = 0$ to $r = 4b$. Is $B(r)$ continuous at $r = a$ and $r = b$? What about its slope?
- (e) Now suppose that a very thin wire running down the center of the conductor carries the same current I in the opposite direction. Can you plot, roughly, the variation of $B(r)$ without another detailed calculation? (Hint: remember that the vectors $d\vec{B}$ from different current elements can be added to obtain the total vector magnetic field.)

9.13.3 Cylinder with a Hole

A long copper rod of radius a has an off-center cylindrical hole through its entire length, as shown in Figure 9.13.3. The conductor carries a current I which is directed out of the page and is uniformly distributed throughout the cross section. Find the magnitude and direction of the magnetic field at the point P .

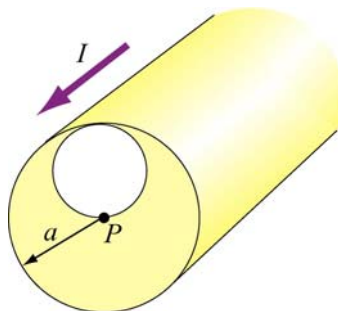


Figure 9.13.3 A cylindrical conductor with a hole.

9.13.4 The Magnetic Field Through a Solenoid

A solenoid has 200 closely spaced turns so that, for most of its length, it may be considered to be an ideal solenoid. It has a length of 0.25 m, a diameter of 0.1 m, and carries a current of 0.30 A.

- Sketch the solenoid, showing clearly the rotation direction of the windings, the current direction, and the magnetic field lines (inside and outside) with arrows to show their direction. What is the dominant direction of the magnetic field inside the solenoid?
- Find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's law.
- Does the magnetic field have a component in the direction of the wire in the loops making up the solenoid? If so, calculate its magnitude both inside and outside the solenoid, at radii 30 mm and 60 mm respectively, and show the directions on your sketch.

9.13.5 Rotating Disk

A circular disk of radius R with uniform charge density σ rotates with an angular speed ω . Show that the magnetic field at the center of the disk is

$$B = \frac{1}{2} \mu_0 \sigma \omega R$$

Hint: Consider a circular ring of radius r and thickness dr . Show that the current in this element is $dI = (\omega / 2\pi) dq = \omega \sigma r dr$.

9.13.6 Four Long Conducting Wires

Four infinitely long parallel wires carrying equal current I are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in Figure 9.13.5. Currents in A and D point out of the page, and into the page at B and C . What is the magnetic field at the center of the square?

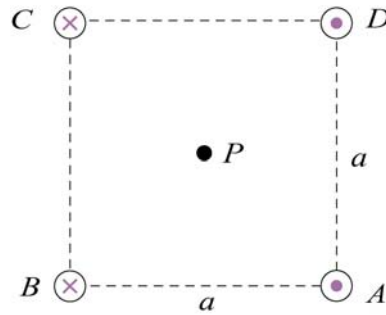


Figure 9.13.5 Four parallel conducting wires

9.13.7 Magnetic Force on a Current Loop

A rectangular loop of length l and width w carries a steady current I_1 . The loop is then placed near a finitely long wire carrying a current I_2 , as shown in Figure 9.13.6. What is the magnetic force experienced by the loop due to the magnetic field of the wire?

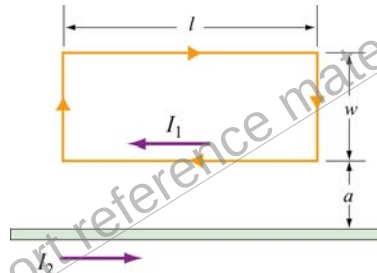


Figure 9.13.6 Magnetic force on a current loop.

9.13.8 Magnetic Moment of an Orbital Electron

We want to estimate the magnetic dipole moment associated with the motion of an electron as it orbits a proton. We use a “semi-classical” model to do this. Assume that the electron has speed v and orbits a proton (assumed to be very massive) located at the origin. The electron is moving in a right-handed sense with respect to the z -axis in a circle of radius $r = 0.53 \text{ \AA}$, as shown in Figure 9.13.7. Note that $1 \text{ \AA} = 10^{-10} \text{ m}$.

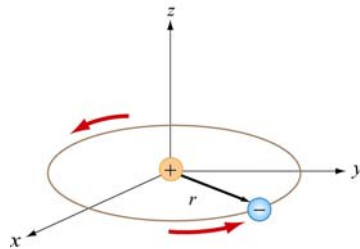


Figure 9.13.7

- (a) The inward force $m_e v^2 / r$ required to make the electron move in this circle is provided by the Coulomb attractive force between the electron and proton (m_e is the mass of the electron). Using this fact, and the value of r we give above, find the speed of the electron in our “semi-classical” model. [Ans: 2.18×10^6 m/s.]
- (b) Given this speed, what is the orbital period T of the electron? [Ans: 1.52×10^{-16} s.]
- (c) What current is associated with this motion? Think of the electron as stretched out uniformly around the circumference of the circle. In a time T , the total amount of charge q that passes an observer at a point on the circle is just e [Ans: 1.05 mA. Big!]
- (d) What is the magnetic dipole moment associated with this orbital motion? Give the magnitude and direction. The magnitude of this dipole moment is one *Bohr magneton*, μ_B . [Ans: 9.27×10^{-24} A·m² along the $-z$ axis.]
- (e) One of the reasons this model is “semi-classical” is because classically there is no reason for the radius of the orbit above to assume the specific value we have given. The value of r is determined from quantum mechanical considerations, to wit that the orbital angular momentum of the electron can only assume integral multiples of $h/2\pi$, where $h = 6.63 \times 10^{-34}$ J/s is the Planck constant. What is the orbital angular momentum of the electron here, in units of $h/2\pi$?

9.13.9 Ferromagnetism and Permanent Magnets

A disk of iron has a height $h = 1.00$ mm and a radius $r = 1.00$ cm. The magnetic dipole moment of an atom of iron is $\mu = 1.8 \times 10^{-23}$ A·m². The molar mass of iron is 55.85 g, and its density is 7.9 g/cm³. Assume that all the iron atoms in the disk have their dipole moments aligned with the axis of the disk.

- (a) What is the number density of the iron atoms? How many atoms are in this disk? [Ans: 8.5×10^{28} atoms/m³; 2.7×10^{22} atoms.]
- (b) What is the magnetization $\vec{M}$ in this disk? [Ans: 1.53×10^6 A/m, parallel to axis.]
- (c) What is the magnetic dipole moment of the disk? [Ans: 0.48 A·m².]
- (d) If we were to wrap one loop of wire around a circle of the same radius r , how much current would the wire have to carry to get the dipole moment in (c)? This is the “equivalent” surface current due to the atomic currents in the interior of the magnet. [Ans: 1525 A.]

9.13.10 Charge in a Magnetic Field

A coil of radius R with its symmetric axis along the $+x$ -direction carries a steady current I . A positive charge q moves with a velocity $\vec{v} = v\hat{j}$ when it crosses the axis at a distance x from the center of the coil, as shown in Figure 9.13.8.

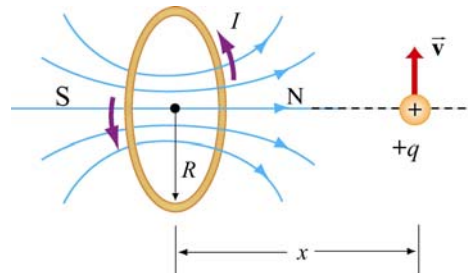


Figure 9.13.8

Describe the subsequent motion of the charge. What is the instantaneous radius of curvature?

9.13.11 Permanent Magnets

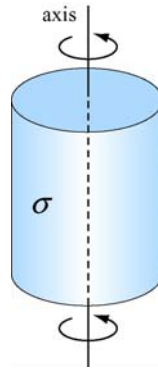
A magnet in the shape of a cylindrical rod has a length of 4.8 cm and a diameter of 1.1 cm. It has a uniform magnetization M of 5300 A/m, directed parallel to its axis.

- (a) Calculate the magnetic dipole moment of this magnet.
- (b) What is the axial field a distance of 1 meter from the center of this magnet, along its axis? [Ans: (a) $2.42 \times 10^{-2} \text{ A} \cdot \text{m}^2$, (b) $4.8 \times 10^{-9} \text{ T}$, or $4.8 \times 10^{-5} \text{ gauss}$.]

9.13.12 Magnetic Field of a Solenoid

- (a) A 3000-turn solenoid has a length of 60 cm and a diameter of 8 cm. If this solenoid carries a current of 5.0 A, find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's Law. How does this compare to the magnetic field of the earth (0.5 gauss). [Ans: 0.0314 T, or 314 gauss, or about 600 times the magnetic field of the earth].

We make a magnetic field in the following way: We have a long cylindrical shell of non-conducting material which carries a surface charge fixed in place (glued down) of $\sigma \text{ C/m}^2$, as shown in Figure 9.13.9 The cylinder is suspended in a manner such that it is free to revolve about its axis, without friction. Initially it is at rest. We come along and spin it up until the speed of the surface of the cylinder is v_0 .

**Figure 9.13.9**

- (b) What is the surface current K on the walls of the cylinder, in A/m? [Ans: $K = \sigma v_0$.]
- (c) What is magnetic field inside the cylinder? [Ans. $B = \mu_0 K = \mu_0 \sigma v_0$, oriented along axis right-handed with respect to spin.]
- (d) What is the magnetic field outside of the cylinder? Assume that the cylinder is infinitely long. [Ans: 0].

9.13.13 Effect of Paramagnetism

A solenoid with 16 turns/cm carries a current of 1.3 A.

- (a) By how much does the magnetic field inside the solenoid increase when a close-fitting chromium rod is inserted? [Note: Chromium is a paramagnetic material with magnetic susceptibility $\chi = 2.7 \times 10^{-4}$.]
- (b) Find the magnitude of the magnetization $\vec{M}$ of the rod. [Ans: (a) $0.86 \mu\text{T}$; (b) 0.68 A/m .]

Magnetic flux and Gauss law

Magnetic Flux

Magnetic flux is a measure of the amount of magnetic field passing through a given area. It is represented by the symbol Φ and has units of webers (Wb) in the International System of Units (SI). Magnetic flux is defined mathematically as the integral of the magnetic field B over an area A :

$$\Phi_B = \mathbf{B} \cdot \mathbf{A} = BA \cos \theta$$

$$\Phi = \int \mathbf{B} \cdot d\mathbf{A}$$

where the dot ($\cdot$) represents the dot product between the magnetic field vector B and the area vector dA .

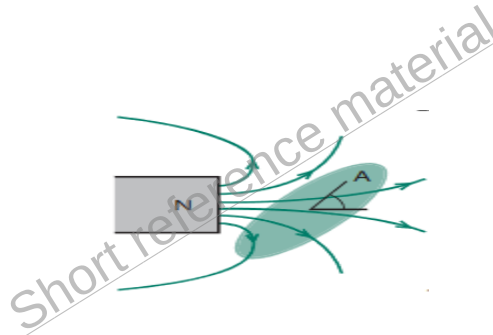


Figure: Area of flux

Magnetic flux is represented by magnetic field lines in diagrams that show magnetic fields. Magnetic flux is an important concept in electromagnetism and is used to describe phenomena such as electromagnetic induction and the behavior of magnetic materials. It is also related to other important quantities, such as magnetic flux density (B), magnetic field strength (H), and magnetic permeability (μ).

Example 1: The dimensions of a square loop is 0.40m x 0.40m. B and θ are 0.05T and 60° respectively. Determine the magnetic flux.

Solution:

$$B = 0.05 \text{ T},$$

$$A = 0.4 \times 0.4 = 0.16 \text{ m}^2,$$

$$\Theta = 60^\circ$$

$$\Phi_B = ?$$

By applying the formula,

$$\begin{aligned}\Phi_B &= BA \cos \theta = 0.05 \times 0.16 \times \cos 60^\circ \\ &= 0.004 \text{ Wb.}\end{aligned}$$

Example 2: A magnetic field of 8.9 T passes perpendicular to a disc with a radius of 5 cm. Find the magnetic flux of the disc.

Solution:

Magnetic field is given by, $B = 8.9 \text{ T}$,

Radius is given by, $r = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$,

As the magnetic field is perpendicular to the disc, $\theta = 0$,

Area is given by $= \pi(5 \times 10^{-2})^2$

$$\Phi_B = ?$$

By applying the formula,

$$\begin{aligned}\Phi_B &= BA \cos \theta = 8.9 \times \pi(5 \times 10^{-2})^2 \times \cos 0 \\ &= 139.7 \times 10^{-4} \text{ Wb.}\end{aligned}$$

Magnetic Flux Linkage

Magnetic flux linkage is a measure of the total amount of magnetic flux that passes through a given coil or circuit. It is represented by the symbol Λ and is measured in units of webers (Wb). Magnetic flux linkage is defined mathematically as the product of the number of turns N in a coil and the magnetic flux Φ that passes through each turn of the coil:

$$\Lambda = N\Phi$$

The concept of magnetic flux linkage is important in the study of electromagnetism and is used to describe phenomena such as electromagnetic induction and the behavior of transformers. In electromagnetic induction, a changing magnetic flux linkage induces an electromotive force (EMF) in a circuit, which in turn can lead to the production of an electric current. In transformers, the ratio of the magnetic flux linkage in the primary and secondary coils determines the voltage and current ratio of the transformer.

Example 1: A coil of wire has 500 turns and is placed in a magnetic field of 0.5 T. If the area of the coil is 0.02 m^2 , what is the magnetic flux linkage?

Solution:

The magnetic flux through each turn is given by:

$$\Phi = B \times A = 0.5 \text{ T} \times 0.02 \text{ m}^2 = 0.01 \text{ Wb}$$

The magnetic flux linkage is given by:

$$\Lambda = N\Phi = 500 \text{ turns} \times 0.01 \text{ Wb} = 5 \text{ Wb}$$

Therefore, the magnetic flux linkage is 5 Wb.

Example 2: A transformer has a primary coil with 100 turns and a secondary coil with 200 turns. If the magnetic flux through the primary coil is 0.02 Wb, what is the magnetic flux linkage in the secondary coil?

Solution:

The magnetic flux linkage in the primary coil is given by:

$$\Lambda_p = N_p \Phi_p = 100 \text{ turns} \times 0.02 \text{ Wb} = 2 \text{ Wb}$$

The transformer is designed such that the magnetic flux is conserved, which means that the magnetic flux linkage in the primary coil is equal to the magnetic flux linkage in the secondary coil:

$$\Lambda_p = \Lambda_s$$

Therefore, the magnetic flux linkage in the secondary coil is:

$$\Lambda_s = \Lambda_p / N_s = 2 \text{ Wb} / 200 \text{ turns} = 0.01 \text{ Wb/turn}$$

Therefore, the magnetic flux linkage in the secondary coil is 0.01 Wb/turn.

Note that in a transformer, the ratio of the number of turns in the primary and secondary coils is equal to the ratio of the voltages in the two coils, as per Faraday's law of electromagnetic induction.

Gauss law in Magnetism

Gauss's law in magnetism, also known as the Gauss's flux law for magnetism, states that the magnetic flux through any closed surface is always zero. Mathematically, it is expressed as:

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

where $\oint$ denotes a closed surface integral, $\mathbf{B}$ is the magnetic field, and $d\mathbf{A}$ is the vector area element of the surface.

This law is a consequence of the fact that there are no magnetic monopoles, which means that magnetic field lines always form closed loops. The magnetic field lines that enter a closed surface must also leave the surface, resulting in zero net flux.

Here are some examples of how Gauss's law in magnetism can be applied:

Example 1: A long straight wire carries a current of 5 A. What is the magnetic field at a distance of 2 cm from the wire?

Solution:

The magnetic field due to a long straight wire can be calculated using Ampere's law. However, we can also use Gauss's law in magnetism to solve this problem.

Consider a cylindrical surface of radius 2 cm and length L that is centered on the wire. The wire passes through the center of the cylinder, and the ends of the cylinder lie on planes perpendicular to the wire.

According to Gauss's law in magnetism, the magnetic flux through the closed surface of the cylinder is zero. The magnetic field is perpendicular to the end faces of the cylinder, so the magnetic flux through the end faces is zero. The magnetic field is parallel to the cylindrical surface, so the magnetic flux through the curved surface is also zero.

Therefore, we can conclude that the magnetic field at a distance of 2 cm from the wire is zero.

Example 2: A long solenoid has 500 turns per meter and carries a current of 3 A. What is the magnetic field inside the solenoid?

Solution:

The magnetic field inside a long solenoid is given by:

$$B = \mu_0 n I$$

where μ_0 is the permeability of free space, n is the number of turns per unit length, and I is the current.

However, we can also use Gauss's law in magnetism to derive this expression.

Consider a cylindrical surface that is coaxial with the solenoid and of radius R . The surface encloses a length L of the solenoid, and it has a top and bottom surface of area A .

According to Gauss's law in magnetism, the magnetic flux through the closed surface is zero. The magnetic field is parallel to the curved surface of the cylinder, so the magnetic flux through the curved surface is zero. The magnetic field is perpendicular to the top and bottom surfaces, so the magnetic flux through these surfaces is $B \cdot A$.

Therefore, we can conclude that the magnetic field inside the solenoid is:

$$B = (BA)/A = nI \times \mu_0$$

where n is the number of turns per unit length, I is the current, and μ_0 is the permeability of free space.

Using the values given in the problem, we can calculate the magnetic field inside the solenoid:

$$B = 500 \text{ turns/m} \times 3 \text{ A} \times 4\pi \times 10^{-7} \text{ T}\cdot\text{m/A} = 6.28 \times 10^{-4} \text{ T}$$

Therefore, the magnetic field inside the solenoid is $6.28 \times 10^{-4} \text{ T}$.

Faraday's Law of Induction

Faraday's law of electromagnetic induction states that a changing magnetic field induces an electromotive force (EMF) in a conductor. The magnitude of the induced EMF is proportional to the rate of change of magnetic flux through the conductor. Mathematically, it can be expressed as:

$$\text{EMF} = -d\Phi/dt$$

where EMF is the electromotive force, Φ is the magnetic flux through the conductor, and t is time.

Lenz's law is a consequence of Faraday's law and states that the direction of the induced EMF and the induced current will be such that it opposes the change that produced it. Lenz's law can be summarized as "nature abhors a change in flux."

Let's take a look at some examples to better understand Faraday's law and Lenz's law:

Example 1: A wire loop with an area of 0.1 m^2 is placed in a uniform magnetic field of 0.5 T . If the loop is rotated from a position where its plane is perpendicular to the magnetic field to a position where it makes an angle of 30 degrees with the field in 0.1 seconds, what is the induced EMF in the loop?

Solution:

The magnetic flux through the wire loop is given by:

$$\Phi = BA\cos(\theta)$$

where B is the magnetic field, A is the area of the loop, and θ is the angle between the loop and the magnetic field.

The rate of change of magnetic flux is given by:

$$d\Phi/dt = -BA\sin(\theta) \times (d\theta/dt)$$

Using the values given in the problem, we can calculate the induced EMF in the loop:

$$d\Phi/dt = -0.5 \text{ T} \times 0.1 \text{ m}^2 \times \sin(30 \text{ degrees}) \times (1/0.1 \text{ s}) = -0.25 \text{ V}$$

Therefore, the induced EMF in the loop is 0.25 V. Note that the negative sign indicates that the induced EMF opposes the change in flux, which is consistent with Lenz's law.

Example 2: A long solenoid with 200 turns per meter and a radius of 0.02 m is connected to a battery with a voltage of 12 V. If the battery is suddenly disconnected from the solenoid, what is the induced EMF and current in the solenoid?

Solution:

When the battery is connected to the solenoid, a magnetic field is produced inside the solenoid. When the battery is suddenly disconnected, the magnetic field inside the solenoid begins to decrease, which induces an EMF in the solenoid. We can use Faraday's law to calculate the induced EMF:

$$\text{EMF} = -d\Phi/dt$$

The magnetic flux through the solenoid is proportional to the current in the solenoid:

$$\Phi = \mu_0 n I A$$

where μ_0 is the permeability of free space, n is the number of turns per unit length, I is the current in the solenoid, and A is the cross-sectional area of the solenoid.

Differentiating this expression with respect to time, we get:

$$d\Phi/dt = \mu_0 n A \times (dI/dt)$$

Using Faraday's law, we can equate the induced EMF to the rate of change of magnetic flux:

$$\text{EMF} = -\mu_0 n A \times (dI/dt)$$

Using the values given in the problem, we can calculate the induced EMF in the solenoid:

$$\text{EMF} = -\mu_0 n A \times (dI/dt) = -4\pi \times 10^{-7} \text{ Tm/A} \times 200 \text{ turns/m} \times \pi \times (0.02 \text{ m}).$$

Example 3: A coil of wire has an area of 0.2 m^2 and is placed in a magnetic field of 0.5 T . The coil is rotated at a frequency of 50 Hz . If the coil has 100 turns, what is the maximum induced emf in the coil?

Solution: The maximum induced emf in the coil is given by the formula:

$$\varepsilon = NBA\omega \sin(\omega t)$$

where N is the number of turns in the coil, B is the magnetic field, A is the area of the coil, ω is the angular frequency of rotation, and t is the time. We are given $N = 100$, $B = 0.5 \text{ T}$, $A = 0.2 \text{ m}^2$, $\omega = 2\pi f = 2\pi \times 50 \text{ Hz} = 314.16 \text{ rad/s}$. We need to find the maximum value of ε .

The maximum value of $\sin(\omega t)$ is 1 , so we can simplify the formula to:

$$\varepsilon = NBA\omega$$

Substituting the given values, we get:

$$\varepsilon = (100)(0.5 \text{ T})(0.2 \text{ m}^2)(314.16 \text{ rad/s}) = 31.4 \text{ V}$$

Therefore, the maximum induced emf in the coil is 31.4 V .

Example 4: A 200 -turn coil of wire is placed in a magnetic field that is changing at a rate of 2 T/s . If the area of the coil is 0.1 m^2 , what is the magnitude of the induced emf in the coil?

Solution: The magnitude of the induced emf in the coil is given by the formula:

$$\varepsilon = -N(d\Phi/dt)$$

where N is the number of turns in the coil and $d\Phi/dt$ is the rate of change of magnetic flux. We are given $N = 200$, $d\Phi/dt = 2 \text{ T/s}$, and $A = 0.1 \text{ m}^2$. We need to find the magnitude of ε .

The magnetic flux through the coil is given by:

$$\Phi = BA$$

where B is the magnetic field. Since the magnetic field is changing, we need to use the formula for the derivative of the magnetic flux:

$$d\Phi/dt = A(dB/dt)$$

Substituting the given values, we get:

$$d\Phi/dt = (0.1 \text{ m}^2)(2 \text{ T/s}) = 0.2 \text{ Wb/s}$$

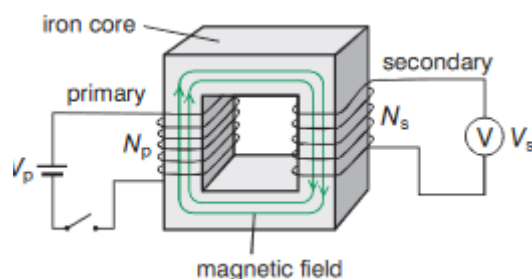
Now we can use the formula for the induced emf:

$$\varepsilon = -N(d\Phi/dt) = -(200)(0.2 \text{ Wb/s}) = -40 \text{ V}$$

Since the induced emf is negative, the direction of the current in the coil will be such that it opposes the change in the magnetic field.

Mutual Inductance and Eddy current

Mutual inductance is the ability of one circuit to induce an electromotive force (EMF) in another circuit through a changing magnetic field. It occurs when two coils of wire are placed near each other so that the magnetic field from one coil passes through the other coil. The amount of mutual inductance between the two coils depends on the number of turns in each coil, the distance between them, and the relative orientation of the coils.



The mutual inductance between two coils can be calculated using the following formula:

$$M = k\sqrt{L_1 L_2}$$

where M is the mutual inductance, L_1 and L_2 are the self-inductances of the two coils, and k is a constant that depends on the relative orientation of the coils.

One application of mutual inductance is in transformers, which are used to change the voltage and current in an electrical circuit. A transformer consists of two coils of wire wrapped around a common core. When an alternating current flows through one coil, it produces a changing magnetic field that induces an EMF in the other coil, which can be used to power another circuit.

Eddy currents are induced currents that flow in conductive materials when they are exposed to a changing magnetic field. They are a form of parasitic current that can cause energy losses and heating in electrical machines, such as motors and transformers. Eddy currents are caused by the interaction between the changing magnetic field and the conductive material, which creates a circular current flow that opposes the original magnetic field.

One way to reduce eddy currents is to use laminated or layered cores made of thin sheets of conductive material, such as iron or steel. The layers are insulated from each other, which reduces the circulation of eddy currents and improves the efficiency of the machine. Another way to reduce eddy currents is to use non-conductive materials, such as ceramics or plastics, in the design of the machine.

Let's take a look at an example of mutual inductance and eddy currents:

Example: Two coils of wire are placed near each other, with 100 turns in each coil and a mutual inductance of 0.5 H. If the current in one coil changes at a rate of 2 A/s, what is the induced EMF in the other coil?

Solution:

The induced EMF in the second coil can be calculated using Faraday's law:

$$\text{EMF} = -M \times (dI/dt)$$

where EMF is the induced EMF, M is the mutual inductance, and dI/dt is the rate of change of current in the first coil.

Using the values given in the problem, we can calculate the induced EMF in the second coil:

$$\text{EMF} = -0.5 \text{ H} \times (2 \text{ A/s}) \times 100 \text{ turns} = -100 \text{ V}$$

Therefore, the induced EMF in the second coil is -100 V.

Now let's consider the example of eddy currents. Suppose a metal plate is placed in a magnetic field that is changing at a rate of 1 T/s. If the plate has a thickness of 1 cm and a resistivity of $2.8 \times 10^{-8} \Omega\text{m}$, what is the magnitude of the eddy currents induced in the plate?

Solution:

The magnitude of the eddy currents can be calculated using the formula:

$$I = \sigma A (dB/dt)$$

where I is the current, σ is the electrical conductivity of the material, A is the area of the plate, and dB/dt is the rate of change of magnetic field.

Using the values given in the problem, we can calculate the magnitude of the eddy currents:

$$I = 2.8 \times 10^{-8} \Omega\text{m} \times (0.01 \text{ m}^2) \times (1 \text{ T/s}) = 2.8 \text{ A}$$

Therefore, the magnitude of the eddy currents induced in the plate is 2.8 A.

To reduce the effect of eddy currents, one can use laminated or layered cores made of thin sheets of conductive material, as mentioned earlier. This is because the thin layers are insulated from each other and therefore the eddy currents will be confined to each layer and not circulate throughout the entire core.

In addition, in electrical machines like motors and transformers, the design of the machine can be modified to reduce eddy currents. For example, using a core made of a non-conductive material, or introducing slots or holes in the core to break up the circulation of eddy currents, can reduce energy losses due to eddy currents.

In conclusion, mutual inductance and eddy currents are important concepts in the study of magnetic fields and their applications in electrical machines. Understanding these concepts and their effects can help in designing more efficient and effective machines.

Self-Inductance

Self-inductance is the ability of a circuit to generate an electromotive force (EMF) in itself due to a changing magnetic field. This occurs when a changing current in a circuit creates a changing magnetic field that induces an EMF in the same circuit. The amount of self-inductance in a circuit is measured in henrys (H) and depends on the geometry of the circuit and the properties of the material.

The self-inductance of a circuit can be calculated using the following formula:

$$L = (N^2 \times \mu \times A) / l$$

where L is the self-inductance, N is the number of turns in the coil, μ is the permeability of the core material, A is the cross-sectional area of the core, and l is the length of the core.

Back EMF (electromotive force) is a phenomenon that occurs in electric motors and generators, where the motor or generator produces a voltage that opposes the applied voltage. It is caused by the interaction between the changing magnetic field and the conductors in the motor or generator, which creates a voltage that opposes the applied voltage.

The magnitude of the back EMF is proportional to the rate of change of the magnetic field and the number of turns in the coil. The back EMF can be calculated using the following formula:

$$\text{EMF} = -L \times (dI/dt)$$

where EMF is the back EMF, L is the self-inductance of the circuit, and dI/dt is the rate of change of current in the circuit.

Let's take a look at an example of self-inductance and back EMF:

Example: A coil with 100 turns and a self-inductance of 0.1 H is connected to a battery with a voltage of 10 V. If the current in the coil changes at a rate of 2 A/s, what is the back EMF generated in the coil?

Solution:

Using the formula for back EMF, we can calculate the back EMF generated in the coil:

$$\text{EMF} = -L \times (dI/dt) = -0.1 \text{ H} \times (2 \text{ A/s}) \times 100 \text{ turns} = -20 \text{ V}$$

Therefore, the back EMF generated in the coil is -20 V.

The effect of back EMF is important in electric motors, as it opposes the applied voltage and reduces the net voltage across the motor. This reduction in voltage causes the motor to draw less current, which reduces the power consumed by the motor and increases its efficiency. In addition, the back EMF can be used to measure the speed of the motor, as the magnitude of the back EMF is proportional to the speed of the motor.

In conclusion, self-inductance and back EMF are important concepts in the study of electrical circuits and motors.

Factors which determines the magnitude of self-inductance and Mutual inductance

The magnitude of self-inductance and mutual inductance depends on several factors, including the geometry of the circuit, the properties of the materials, and the number of turns in the coil.

Factors that determine the magnitude of self-inductance:

1. Number of turns: The self-inductance of a coil is directly proportional to the number of turns in the coil. Increasing the number of turns in the coil increases the self-inductance.
2. Area of the coil: The self-inductance of a coil is directly proportional to the cross-sectional area of the coil. Increasing the area of the coil increases the self-inductance.
3. Length of the coil: The self-inductance of a coil is inversely proportional to the length of the coil. Increasing the length of the coil decreases the self-inductance.
4. Permeability of the core material: The self-inductance of a coil is directly proportional to the permeability of the core material. Increasing the permeability of the core material increases the self-inductance.

Factors that determine the magnitude of mutual inductance:

1. Number of turns: The mutual inductance between two coils is directly proportional to the number of turns in each coil. Increasing the number of turns in either or both coils increases the mutual inductance.
2. Distance between the coils: The mutual inductance between two coils is inversely proportional to the distance between the coils. Decreasing the distance between the coils increases the mutual inductance.
3. Orientation of the coils: The mutual inductance between two coils is dependent on the orientation of the coils with respect to each other. The maximum mutual inductance occurs when the coils are aligned with their axes parallel to each other.
4. Permeability of the core material: The mutual inductance between two coils is directly proportional to the permeability of the core material. Increasing the permeability of the core material increases the mutual inductance.

In conclusion, the magnitude of self-inductance and mutual inductance depends on various factors that can be controlled and manipulated in the design of electrical circuits and devices. Understanding these factors is crucial in optimizing the performance of these systems.

Inductance and Induced EMF

Inductance is a property of an electrical circuit that describes the ability of a circuit to resist changes in current. Inductors are electrical components that are designed to have a specific amount of inductance, which is measured in henries (H). Inductors are made up of coils of wire, and the amount of inductance is determined by the number of turns in the coil, the cross-sectional area of the coil, and the type of core material used.

Induced EMF is the electromotive force that is generated in a circuit as a result of a change in magnetic flux. This phenomenon is described by Faraday's law of induction, which states that the EMF induced in a circuit is proportional to the rate of change of magnetic flux through the circuit.

Example of Inductance and Induced EMF:

Consider a circuit that consists of a coil of wire with an inductance of 10 mH and a resistance of 100 ohms. If a current of 1 ampere is flowing through the coil and the current is suddenly switched off, what is the magnitude of the induced EMF?

The rate of change of current in the coil is given by dI/dt , where I is the current and t is time. When the current is switched off, the rate of change of current is equal to -1 A/s . The inductance of the coil is $L = 10 \text{ mH} = 0.01 \text{ H}$. According to Faraday's law, the induced EMF is given by:

$$\text{EMF} = -L(dI/dt)$$

Substituting the values, we get:

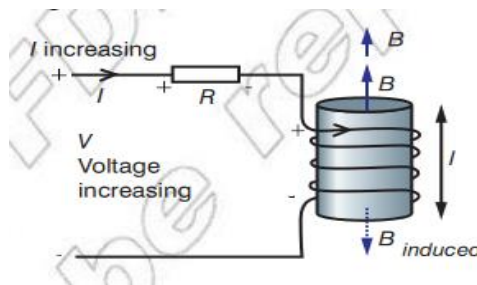
$$\text{EMF} = -0.01 \times (-1) = 0.01 \text{ volts}$$

Therefore, the magnitude of the induced EMF is 0.01 volts.

In conclusion, inductance and induced EMF are important concepts in the study of electrical circuits. Inductance determines the ability of a circuit to resist changes in current, while induced EMF is the electromotive force that is generated in a circuit as a result of a change in magnetic flux. Understanding these concepts is crucial in designing and analyzing electrical circuits and devices.

Inductance of Solenoid

The inductance of a solenoid is determined by the number of turns in the solenoid, the length of the solenoid, the cross-sectional area of the solenoid, and the permeability of the core material. The inductance of a solenoid is given by the equation:



$$L = (\mu \times N^2 \times A) / l$$

Where L is the inductance in henries (H), μ is the permeability of the core material, N is the number of turns in the solenoid, A is the cross-sectional area of the solenoid, and l is the length of the solenoid.

Example: Suppose we have a solenoid with 2000 turns, a length of 10 cm, a radius of 2 cm, and is wound on a ferromagnetic core material with a permeability of 2000μ . What is the inductance of the solenoid?

The cross-sectional area of the solenoid can be calculated using the formula for the area of a circle:

$$A = \pi \times r^2$$

Where A is the cross-sectional area of the solenoid, and r is the radius of the solenoid. Substituting the values, we get:

$$A = \pi \times (0.02m)^2 = 0.0012566 \text{ m}^2.$$

Now, we can calculate the inductance of the solenoid using the formula:

$$L = (\mu \times N^2 \times A) / l$$

Substituting the values, we get:

$$L = (2000 \times 2000 \times 0.0012566) / 0.1$$

$$L = 100.53 \text{ H}$$

Therefore, the inductance of the solenoid is 100.53 H.

In conclusion, the inductance of a solenoid is determined by various factors such as the number of turns, length, cross-sectional area, and permeability of the core material. The inductance of a solenoid is an important parameter in electrical circuits that use solenoids, such as inductors, transformers, and electromagnets.

Magnetic Energy Density

Magnetic energy density is the amount of magnetic energy stored in a given volume of space. It is calculated using the magnetic field strength (B) and the permeability of free space (μ_0). The magnetic energy density (u) is given by:

$$u = (1/2) \times \mu_0 \times B^2$$

Where μ_0 is the permeability of free space, and B is the magnetic field strength.

Example 1:

Suppose we have a solenoid with a radius of 3 cm, 10 turns per cm, and a current of 2 A flowing through it. What is the magnetic energy density inside the solenoid?

To find the magnetic energy density, we need to first find the magnetic field strength inside the solenoid. The magnetic field strength can be calculated using the formula:

$$B = \mu_0 \times N \times I / L$$

Where N is the number of turns per unit length, I is the current, L is the length of the solenoid, and μ_0 is the permeability of free space. Substituting the values, we get:

$$B = (4\pi \times 10^{-7}) \times 10 \times 2 / 0.03$$

$$B = 8.38 \times 10^{-3} \text{ T}$$

Now, we can calculate the magnetic energy density using the formula:

$$u = (1/2) \times \mu_0 \times B^2$$

Substituting the values, we get:

$$u = (1/2) \times (4\pi \times 10^{-7}) \times (8.38 \times 10^{-3})^2$$

$$u = 1.12 \times 10^{-10} \text{ J/m}^3$$

Therefore, the magnetic energy density inside the solenoid is $1.12 \times 10^{-10} \text{ J/m}^3$.

Example 2: A magnetic field has a flux density of 0.5 T and is present in a medium with a permeability of $4\pi \times 10^{-7} \text{ H/m}$. What is the magnetic energy density of the field?

Solution: We are given the flux density $B = 0.5 \text{ T}$ and the permeability $\mu = 4\pi \times 10^{-7} \text{ H/m}$.

Substituting these values in the formula for magnetic energy density, we get:

$$u = (B^2)/(2\mu) = (0.5\text{T})^2 / (2 \times 4\pi \times 10^{-7} \text{ H/m}) = 1.25 \times 10^{-3} \text{ J/m}^3$$

Therefore, the magnetic energy density of the field is $1.25 \times 10^{-3} \text{ J/m}^3$.

We can also calculate the energy stored in the magnetic field by multiplying the magnetic energy density with the volume of the region in which the field is present. For example, if the magnetic field is present in a volume of 1 m^3 , then the total energy stored in the field would be:

$$E = u \times V = (1.25 \times 10^{-3} \text{ J/m}^3) \times (1 \text{ m}^3) = 1.25 \times 10^{-3} \text{ J}$$

This means that $1.25 \times 10^{-3} \text{ J}$ of energy is stored in the magnetic field per cubic meter of volume.

In conclusion, magnetic energy density is a useful parameter for calculating the amount of magnetic energy stored in a given volume of space. It is determined by the magnetic field strength and the permeability of free space.

Alternating current, generator and transformer

Direct current and alternating current

Direct current is an electrical current that flows in one direction only. The voltage and current in a DC circuit do not change direction over time. DC is commonly used in battery-powered devices, such as flashlights, smartphones, and portable radios.

Alternating current, on the other hand, is an electrical current that changes direction periodically. The voltage and current in an AC circuit alternate direction many times per second, typically at a frequency of 50 or 60 Hz. AC is commonly used for power transmission and distribution in electrical grids, as well as in household appliances and electronic devices.

The main difference between DC and AC is the direction and frequency of the current. DC flows in one direction, while AC changes direction periodically. Another key difference is that DC voltage can be constant, while AC voltage varies over time.

In terms of advantages and disadvantages, DC has the advantage of being able to provide a constant voltage, which makes it suitable for powering devices that require a steady voltage

source. However, DC power is difficult to transmit over long distances, and it requires the use of expensive high-voltage DC transmission lines.

AC, on the other hand, can be easily transmitted over long distances using transformers, which makes it the preferred choice for power transmission and distribution. AC power is also easier and cheaper to generate, and it can be used to power a wide variety of devices and appliances. DC and AC are two types of electrical current that have different properties and applications.

Solved examples

Direct Current (DC):

1. A battery with an emf of 12 V is connected to a circuit with a total resistance of 4 ohms. What is the current flowing through the circuit?

Solution:

Using Ohm's Law, we know that the current flowing through a circuit is equal to the voltage divided by the resistance:

$$I = V/R$$

Plugging in the given values, we get:

$$I = 12/4$$

$$I = 3 \text{ amperes}$$

Therefore, the current flowing through the circuit is 3 amperes.

2. An electric motor draws a current of 10 A from a DC source with an emf of 120 V. What is the power consumed by the motor?

Solution:

The power consumed by an electric device can be calculated using the formula:

Power = Current x Voltage

Plugging in the given values, we get:

$$\text{Power} = 10 \times 120$$

$$\text{Power} = 1200 \text{ watts}$$

Therefore, the power consumed by the motor is 1200 watts.

Example 3 An AC circuit has a voltage of 220 V and a frequency of 50 Hz. If the circuit has an inductance of 0.1 H and a resistance of 10 ohms, what is the impedance of the circuit?

Solution:

The impedance of the circuit can be calculated using the formula:

$$Z = \sqrt{R^2 + X^2}$$

where R is the resistance of the circuit, and X is the reactance of the circuit. Reactance is the opposition to the flow of AC caused by inductance or capacitance.

The reactance of the circuit can be calculated using the formula:

$$X = 2 \times \pi \times f \times L$$

where f is the frequency of the AC, and L is the inductance of the circuit.

$$X = 2 \times 3.14 \times 50 \times 0.1$$

$$X = 31.4 \text{ ohms}$$

Now we can calculate the impedance of the circuit:

$$Z = \sqrt{10^2 + 31.4^2}$$

$$Z = 33.1 \text{ ohms}$$

Therefore, the impedance of the circuit is 33.1 ohms.

Example 4: An AC circuit has a voltage of 110 V and a frequency of 60 Hz. The circuit is connected to a 50 ohm resistor and a 0.1 microfarad capacitor. What is the current flowing through the circuit?

Solution:

The reactance of the capacitor can be calculated using the formula:

$$X_c = 1 / (2 \times \pi \times f \times C)$$

where C is the capacitance of the capacitor.

$$X_c = 1 / (2 \times 3.14 \times 60 \times 0.1 \times 10^{-6})$$

$$X_c = 265.2 \text{ ohms}$$

Now we can calculate the total impedance of the circuit:

$$Z = \sqrt{R^2 + X^2}$$

$$Z = \sqrt{50^2 + 265.2^2}$$

$$Z = 289.4 \text{ ohms}$$

Using Ohm's law, we can find the current flowing through the circuit:

$$I = V/Z$$

$$I = 110/289.4$$

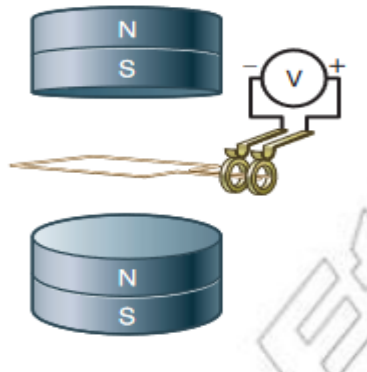
$$I = 0.38 \text{ A}$$

Therefore, the current flowing through the circuit is 0.38 A.

Electric Generator

An electric generator is a device that converts mechanical energy into electrical energy. It works on the principle of electromagnetic induction, where a conductor is moved through a

magnetic field to produce an electric current. The basic components of an electric generator are a rotor, a stator, and a source of mechanical energy.



Example:

Suppose we have an electric generator with a rotor of 0.5 m radius that rotates at a speed of 1200 rpm. The magnetic field strength is 0.3 T, and the generator has 200 turns of wire in the stator coil. What is the voltage generated by the generator?

To find the voltage generated by the generator, we need to use the formula:

$$\text{EMF} = N \times \Delta\Phi / \Delta t$$

Where EMF is the electromotive force or voltage, N is the number of turns in the coil, $\Delta\Phi$ is the change in magnetic flux, and Δt is the time it takes for the change to occur.

The change in magnetic flux can be calculated using the formula:

$$\Delta\Phi = B \times A \times \Delta\theta$$

Where B is the magnetic field strength, A is the area of the coil, and $\Delta\theta$ is the change in angle between the magnetic field and the coil.

Substituting the values, we get:

$$A = \pi \times r^2 = \pi \times 0.5^2 = 0.785 \text{ m}^2 \quad \Delta\theta = (2\pi / 60) \times 1200 = 40\pi \text{ rad/s} \quad \Delta\Phi = 0.3 \times 0.785 \times 40\pi = 37.7 \text{ Wb}$$

Now we can calculate the EMF using the formula:

$$\text{EMF} = N \times \Delta\Phi / \Delta t$$

Δt is the time it takes for one revolution, which can be calculated as:

$$\Delta t = 1 / (1200 / 60) = 0.05 \text{ s}$$

Substituting the values, we get:

$$\text{EMF} = 200 \times 37.7 / 0.05 = 15080 \text{ V}$$

Therefore, the voltage generated by the generator is 15080 V.

In conclusion, an electric generator is a device that converts mechanical energy into electrical energy using the principle of electromagnetic induction. The voltage generated by the generator can be calculated using the formula

$$\text{EMF} = N \times \Delta\Phi / \Delta t,$$

where N is the number of turns in the coil, $\Delta\Phi$ is the change in magnetic flux, and Δt is the time it takes for the change to occur.

Solved examples

1. A generator produces an AC voltage of 120 V with a frequency of 60 Hz. If the number of poles in the generator is 4, what is the rotational speed of the generator?

Solution:

The frequency of the AC voltage produced by the generator is related to the rotational speed and the number of poles by the formula:

$$f = (P \times N) / 120$$

where f is the frequency in Hz, P is the number of poles, and N is the rotational speed in revolutions per minute (rpm).

Plugging in the given values, we get:

$$60 = (4 \times N) / 120$$

$$N = (60 \times 120) / 4$$

$$N = 1800 \text{ rpm}$$

Therefore, the rotational speed of the generator is 1800 rpm.

2. An electric generator produces a voltage of 240 V with a current of 10 A. If the resistance of the load connected to the generator is 20 ohms, what is the power output of the generator?

Solution:

The power output of the generator is given by the formula:

$$\text{Power} = \text{Voltage} \times \text{Current}$$

Plugging in the given values, we get:

$$\text{Power} = 240 \times 10$$

$$\text{Power} = 2400 \text{ W}$$

Since the load connected to the generator has a resistance of 20 ohms, we can also use the formula:

$$\text{Power} = \text{Current}^2 \times \text{Resistance}$$

Plugging in the given values, we get:

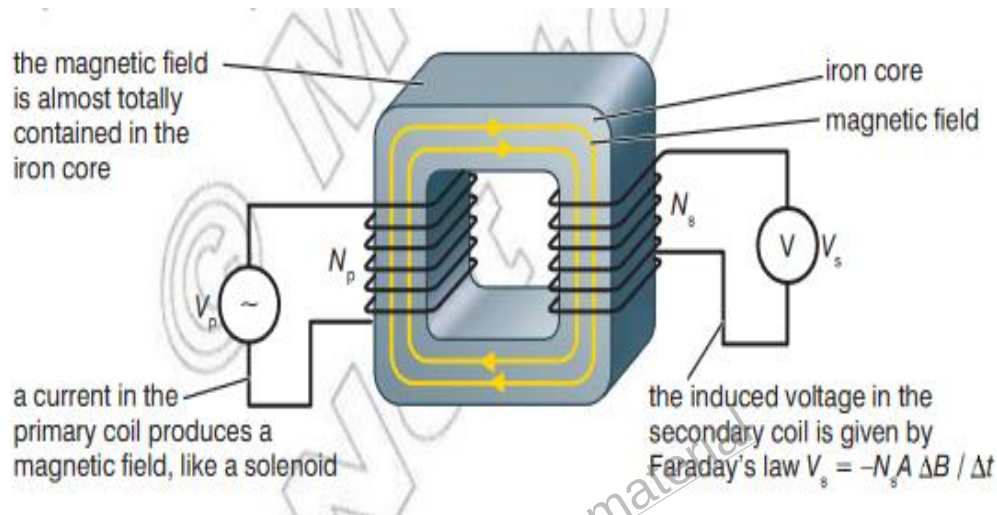
$$\text{Power} = 10^2 \times 20$$

$$\text{Power} = 2000 \text{ W}$$

The two answers are not the same because the voltage and current output of the generator may vary depending on the load connected to it. However, both formulas can be used to calculate the power output of the generator under different conditions.

Transformer

A transformer is an electrical device that is used to transfer electrical energy from one circuit to another through electromagnetic induction. It consists of two coils, known as the primary coil and the secondary coil, which are wrapped around a common magnetic core. The primary coil is connected to an alternating current source, and the secondary coil is connected to a load. When the alternating current flows through the primary coil, it generates a magnetic field that induces a voltage in the secondary coil, which is then used to power the load.



The principle of operation of a transformer can be explained by Faraday's law of electromagnetic induction. When an alternating current flows through the primary coil, it creates a changing magnetic field that links with the secondary coil. This changing magnetic field induces a voltage in the secondary coil, which is proportional to the rate of change of the magnetic field. The magnitude of the voltage induced in the secondary coil is determined by the number of turns in the secondary coil, the rate of change of the magnetic field, and the frequency of the alternating current.

The basic components of a transformer are the core, the primary winding, and the secondary winding. The core is made of a ferromagnetic material, such as iron or steel, which enhances the magnetic coupling between the primary and secondary coils. The primary winding is connected to an alternating current source, which produces the alternating magnetic field. The secondary winding is connected to the load, which uses the voltage induced in the secondary coil to power electrical devices.

Example:

Suppose we have a transformer with 1000 turns in the primary coil and 2000 turns in the secondary coil. The voltage in the primary coil is 240 V, and the load connected to the secondary coil requires a voltage of 120 V. What is the current in the secondary coil?

To solve this problem, we can use the formula:

$$V_p / V_s = N_p / N_s$$

Where V_p is the voltage in the primary coil, V_s is the voltage in the secondary coil, N_p is the number of turns in the primary coil, and N_s is the number of turns in the secondary coil.

Substituting the values, we get:

$$240 / 120 = 1000 / 2000$$

Simplifying the equation, we get:

$$N_s = 2 \times N_p$$

Therefore, the number of turns in the secondary coil is twice the number of turns in the primary coil.

Now, we can use the formula for power in a transformer:

$$P_p = P_s$$

Where P_p is the power in the primary coil and P_s is the power in the secondary coil.

Substituting the values, we get:

$$P_p = V_p \times I_p \quad P_s = V_s \times I_s$$

Since the power is the same in both coils, we can equate the two equations:

$$V_p \times I_p = V_s \times I_s$$

Substituting the values, we get:

$$240 \times I_p = 120 \times I_s$$

Simplifying the equation, we get:

$$I_p = I_s / 2$$

Therefore, the current in the secondary coil is twice the current in the primary coil.

In conclusion, a transformer is an electrical device that is used to transfer electrical energy from one circuit to another through electromagnetic induction. The voltage induced in the secondary coil is proportional to the number of turns in the secondary coil, the rate of change of the magnetic field, and the frequency of the alternating current. The basic components of a transformer are the core, the primary winding, and the secondary winding. Transformers are widely used in power distribution systems to step up or step down the voltage of the electrical power transmitted over long distances.

Here are some more examples of how transformers are used in various applications:

1. **Power transmission:** Transformers are used to step up the voltage of the electrical power generated at power plants to high voltages for transmission over long distances. This reduces the amount of power loss during transmission and enables efficient delivery of electricity to homes and businesses.
2. **Voltage regulation:** Transformers are used in voltage regulators to maintain a constant voltage output, despite fluctuations in the input voltage. This is important in electronic devices that require a stable voltage input to function properly.
3. **Audio amplification:** Transformers are used in audio amplifiers to step up the low-level audio signal from a microphone or guitar pickup to a higher voltage that can be amplified by the amplifier. This enables the amplifier to produce a louder sound without distorting the signal.
4. **Welding:** Transformers are used in welding machines to convert the low voltage and high current from the power source into high voltage and low current needed for welding. This reduces the risk of electric shock and enables more precise control of the welding process.
5. **Lighting:** Transformers are used in halogen lamps to step up the voltage of the electricity flowing through the lamp filament, which increases the brightness of the light produced.

6. Battery charging: Transformers are used in battery chargers to step down the voltage of the power supply to the level required for charging the battery. This prevents damage to the battery and ensures safe and efficient charging.

These are just a few examples of how transformers are used in various applications. The versatility and efficiency of transformers make them a critical component in many electrical systems.

Solved examples

1. A transformer has 100 turns on its primary coil and 500 turns on its secondary coil. If the voltage applied to the primary coil is 120 V, what is the voltage across the secondary coil?

Solution:

We can use the transformer equation to solve this problem:

$$V_p/V_s = N_p/N_s$$

Where V_p is the voltage across the primary coil, V_s is the voltage across the secondary coil, N_p is the number of turns on the primary coil, and N_s is the number of turns on the secondary coil.

Plugging in the given values, we get:

$$120/V_s = 100/500$$

$$V_s = (120 \times 500) / 100$$

$$V_s = 600 \text{ V}$$

Therefore, the voltage across the secondary coil is 600 V.

2. A step-up transformer has 200 turns on its primary coil and 1000 turns on its secondary coil. If the power input to the primary coil is 500 W, what is the power output from the secondary coil?

Solution:

We know that the power input to the primary coil is equal to the power output from the secondary coil (assuming no losses). So we can use the formula:

$$\text{Power} = \text{Voltage} \times \text{Current}$$

And the transformer equation:

$$V_p/V_s = N_p/N_s$$

Where V_p is the voltage across the primary coil, V_s is the voltage across the secondary coil, N_p is the number of turns on the primary coil, and N_s is the number of turns on the secondary coil.

First, we need to find the voltage across the secondary coil. Using the transformer equation, we get:

$$V_p/V_s = N_p/N_s$$

$$120/V_s = 200/1000$$

$$V_s = (120 \times 1000) / 200$$

$$V_s = 600 \text{ V}$$

Now, we can find the current in the secondary coil using:

$$\text{Current} = \text{Power} / \text{Voltage}$$

$$\text{Current} = 500 / 600$$

$$\text{Current} = 0.833 \text{ A}$$

Finally, we can find the power output from the secondary coil using:

$$\text{Power} = \text{Voltage} \times \text{Current}$$

$$\text{Power} = 600 \times 0.833$$

$$\text{Power} = 499.8 \text{ W}$$

Therefore, the power output from the secondary coil is 499.8 W.

Short reference material

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