ODA BOARDING SCHOOL

GRADE 12 MATHEMATICS MULTI TOPICS SOLVED EXAMPLES

1. Find the coefficient of x^{10} in the expansion of $(1+2x)^{-7}$?

Solve: By Binomial series:

$$(1+2x)^{-7} = 1+(-7)(2x)+\frac{(-7)(-7-1)}{2!}(2x)^2+\frac{(-7)(-7-1)(-7-2)}{3!}(2x)^3+\cdots+$$

$$\frac{(-7)(-8)(-9)(-10)(-11)(-12)(-13)(-14)(-15)(-16)}{10!}(2x)^{10}+\cdots,$$

converges for $|2x|<1 \Rightarrow |x|<1/2$.

So the coefficient of x^{10} is $2^{10}\left[\frac{(1)(8)(9)\cdots(16)}{10!}\right] = 2^{10}C(16,10)$.

2. If $A=\begin{pmatrix}4 & 3\\ 2 & 5\end{pmatrix}$ and $A^2-mA+nI=0$, then find the values of m and n.

Solve:

$$A^2-mA+nI = \begin{pmatrix}4 & 3\\ 2 & 5\end{pmatrix}\begin{pmatrix}4 & 3\\ 2 & 5\end{pmatrix}-m\begin{pmatrix}4 & 3\\ 2 & 5\end{pmatrix}+n\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}22 & 27\\ 18 & 31\end{pmatrix}-\begin{pmatrix}4m & 3m\\ 2m & 5m\end{pmatrix}+\begin{pmatrix}n & 0\\ 0 & n\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow \begin{cases}22-4m+n=0\\ 27-3m=0 \Rightarrow m=9 \text{ and } n=14\end{cases}$$

$\therefore m=9$ and $n=14$.

3. If $f(a)=\begin{vmatrix}1 & a & a^2\\ a & a^2 & 1\\ a^2 & 1 & a\end{vmatrix}$, then what is the value of $f(\sqrt[3]{3})$?

Solve:

$$f(a) = 1\begin{vmatrix}a^2 & 1\\ 1 & a\end{vmatrix}-a\begin{vmatrix}a & 1\\ a^2 & a\end{vmatrix}+a^2\begin{vmatrix}a & a^2\\ a^2 & 1\end{vmatrix}$$

$$= a^3-1-a(a^2-a^2)+a^2(a-a^4)$$

$$= a^3-1-0+a^3-a^6$$

$$\Rightarrow f(a) = 2a^3-a^6-1$$

So, $f(\sqrt[3]{3}) = f(3^{1/3}) = 2(3^{1/3})^3-(3^{1/3})^6-1$

$$= 2(3^{3/3})-3^{6/3}-1$$

$$\therefore f(\sqrt[3]{3}) = 6-9-1$$

$$\therefore f(\sqrt[3]{3}) = -4$$

4. If $z = x+iy$, $z^{1/3} = a-ib$ and $\frac{x}{a} - \frac{y}{b} = A(a^2+b^2)$, then find A.

Sol:- $z^{1/3} = a-ib \Rightarrow (x+iy)^{1/3} = a-ib$

$\Rightarrow x+iy = (a-ib)^3$

$\Rightarrow x+iy = a^3 + ib^3 - 3a^2bi - 3ab^2$

$\Rightarrow x+iy = a^3 - 3ab^2 + (b^3 - 3a^2b)i$

$\Rightarrow x = a^3 - 3ab^2$ and $iy = b^3 - 3a^2bi$

$\Rightarrow x = a(a^2 - 3b^2)$ and $y = b(b^2 - 3a^2)$

$\Rightarrow \frac{x}{a} = (a^2 - 3b^2)$ and $\frac{y}{b} = b^2 - 3a^2$

Now, $\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - (b^2 - 3a^2)$

$= a^2 - 3b^2 - b^2 + 3a^2$

$= 4a^2 - 4b^2$

$\Rightarrow \frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2)$

Therefore $A = 4$ //

5. If $x = -3+5i$, then what is the value of $x^3 + 6x^2 + 34x + 1$?

Sol:- $x = -3+5i \Rightarrow x+3 = 5i$

$\Rightarrow (x+3)^2 = (5i)^2$ ... Squaring both sides

$\Rightarrow x^2 + 6x + 9 = 25i^2$ but $i^2 = -1$

$\Rightarrow x^2 + 6x + 9 = -25$

$\Rightarrow x^2 + 6x + 34 = 0$

Therefore, $x^3 + 6x^2 + 34x + 1 = x(x^2 + 6x + 34) + 1$ but $x^2 + 6x + 34 = 0$

$= x(0) + 1$

$= 0 + 1$

$= 1$ //

6. If $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$, $0<\alpha, \beta<\frac{\pi}{4}$, then find the value of $\tan 2\alpha$.

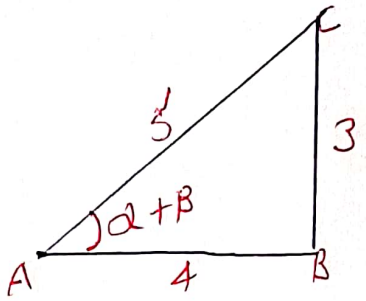
Solve: $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$, $0<\alpha, \beta<\frac{\pi}{4}$.

$\tan 2\alpha = \tan(\alpha+\beta+\alpha-\beta)$.

Remember that $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$.

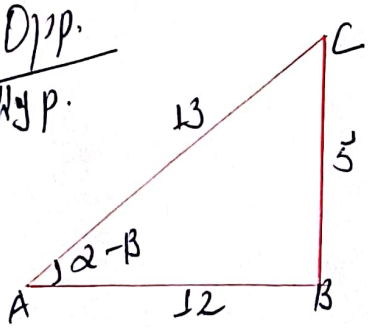
So, $\tan((\alpha+\beta)+(\alpha-\beta))=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$.

And, $\cos(\alpha+\beta)=\frac{4}{5}=\frac{\text{Adjacent}}{\text{Hypothenuse}}$



$\Rightarrow \sin(\alpha+\beta)=\frac{3}{5}$.

$\sin(\alpha-\beta)=\frac{5}{13}=\frac{\text{Opp.}}{\text{Hyp.}}$



$\Rightarrow \cos(\alpha-\beta)=\frac{12}{13}$

Therefore, $\tan 2\alpha=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$

$$=\frac{\frac{3/5}{4/5}+\frac{5/13}{12/13}}{1-\left(\frac{3}{4}\right)\left(\frac{5}{12}\right)}$$

$$=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{15}{48}}=\frac{9+5}{12}\times\frac{48}{33}=\frac{56}{33}$$

$\Rightarrow \tan 2\alpha=\frac{56}{33}$ //

7. Find the no of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$.

Sol: Let $m = \sin x$

So, $2\sin^2 x + 5\sin x - 3 = 0 \Rightarrow 2m^2 + 5m - 3 = 0$

$\Rightarrow (2m-1)(m+3) = 0$

$\Rightarrow 2m - 1 = 0$ or $m + 3 = 0$

$\Rightarrow m = 1/2$ or $m = -3$ but $m = \sin x$

$\Rightarrow \sin x = 1/2$ or $\sin x = -3$ not possible

$\therefore \sin x = 1/2$, $x \in [0, 3\pi]$

$\therefore$ Two solutions are possible in $[0, \pi]$ and two solutions are possible in $[2\pi, 3\pi]$.

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ are possible solutions.

$\therefore$ The no of solutions is 4. //

8. If $\sin\alpha - \sin\beta = m$ and $\cos\alpha - \cos\beta = n$, then find $\cos(\alpha - \beta)$ in terms of $m+n$.

Sol: $m = \sin\alpha - \sin\beta \Rightarrow m^2 = \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$ and

$n = \cos\alpha - \cos\beta \Rightarrow n^2 = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta$

Now, $m^2 + n^2 = (\sin^2\alpha + \cos^2\alpha) - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta + (\sin^2\beta + \cos^2\beta)$

$= 1 - 2(\sin\alpha\sin\beta + \cos\alpha\cos\beta) + 1$

$= 2 - 2(\cos(\alpha - \beta))$

$\Rightarrow m^2 + n^2 - 2 = -2\cos(\alpha - \beta)$

$\Rightarrow \cos(\alpha - \beta) = \frac{m^2 + n^2 - 2}{-2}$

$\therefore \cos(\alpha - \beta) = \frac{2 - m^2 - n^2}{2}$ //

9. Find the value of $\frac{\tan 80^0 - \tan 10^0}{\tan 70^0}$

Sol:- Remember that $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$.

Then $\tan 70^0 = \tan(80^0 - 10^0)$

$\Rightarrow \tan 70^0 = \frac{\tan 80^0 - \tan 10^0}{1+\tan 80^0 \tan 10^0}$

$\Rightarrow \tan 70^0(1+\tan 80^0 \tan 10^0) = \tan 80^0 - \tan 10$

Then $\frac{\tan 80^0 - \tan 10^0}{\tan 70^0} = \frac{\tan 70^0(1+\tan 80^0 \tan 10^0)}{\tan 70^0}$, dividing both sides by $\tan 70^0$

$\Rightarrow \frac{\tan 80^0 - \tan 10}{\tan 70^0} = 1+\tan 10^0 \tan 80^0$ But $\tan 80^0 = \cot 10^0$

$= 1+\tan 10^0 \cot 10^0$

$= 1 + \frac{\sin 10^0}{\cos 10^0} \times \frac{\cos 10^0}{\sin 10^0}$

$= 1+1$

$= 2$

$\therefore \frac{\tan 80^0 - \tan 10^0}{\tan 70^0} = 2$ //

10. If $x = \tan 10^0$, then find $\tan 70^0$ in terms of x.

Sol:- Remember that $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Then $\tan 20^0 = \tan 2(10^0) = \frac{2\tan 10^0}{1-\tan^2(10^0)} = \frac{2x}{1-x^2}$

So, $\tan 70^0 = \cot 20^0$

$= \cos 2(10^0)$

$= \frac{1}{\tan 2(10^0)} = \frac{1}{2x/1-x^2} = \frac{1-x^2}{2x}$

$\therefore \tan 70^0 = \frac{1-x^2}{2x}$ //

11. Find $\int \sec^2 x \csc^2 x \, dx$

Sol:

$$\int \sec^2 x \csc^2 x \, dx = \int \frac{1}{\cos^2 x} \cdot \frac{1}{\sin^2 x} \, dx$$

$$= \int \frac{1}{\cos^2 x \sin^2 x} \, dx \quad \text{but } 1 = \cos^2 x + \sin^2 x$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \, dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} \, dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} \, dx$$

$$= \int \frac{1}{\cos^2 x} \, dx + \int \frac{1}{\sin^2 x} \, dx$$

$$= \int \sec^2 x \, dx + \int \csc^2 x \, dx$$

$$\therefore \int \sec^2 x \csc^2 x \, dx = \tan x - \cot x + C //$$

12. If the radius of the sphere is doubled, then its volume becomes:

A. double B. 4 times C. 6 times D. 8 times.

Sol:

- Volume of the old sphere $= \frac{4}{3}\pi r^3$
- Volume of the new sphere $= \frac{4}{3}\pi R^3$, R is radius of the new sphere & $R = 2r$.

$$= \frac{4}{3}\pi (2r)^3$$

$$= \frac{4}{3}\pi (8r^3)$$

$$= 8\left[\frac{4}{3}\pi r^3\right]$$

= 8 times the volume of the old one.

Answer: D

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